

# Strings in $\text{AdS}_4 \times \mathbb{CP}^3$ : finite size spectrum vs. Bethe Ansatz

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**ABSTRACT:** We compute the first curvature corrections to the spectrum of light-cone gauge type IIA string theory that arise in the expansion of  $\text{AdS}_4 \times \mathbb{CP}^3$  about a plane-wave limit. The resulting spectrum is shown to match precisely, both in magnitude and degeneration that of the corresponding solutions of the all-loop Gromov–Vieira Bethe Ansatz. The one-loop dispersion relation correction is calculated for all the single oscillator states of the theory, with the level matching condition lifted. It is shown to have all logarithmic divergences cancelled and to leave only a finite exponentially suppressed contribution, as shown earlier for light bosons. We argue that there is no ambiguity in the choice of the regularization for the self-energy sum, since the regularization applied is the only one preserving unitarity. Interaction matrices in the full degenerate two-oscillator sector are calculated and the spectrum of all two light magnon oscillators is completely determined. The same finite-size corrections, at the order  $\frac{1}{J}$ , where  $J$  is the length of the chain, in the two-magnon sector are calculated from the all loop Bethe Ansatz. The corrections obtained by the two completely different methods coincide up to the fourth order in  $\lambda' \equiv \frac{\lambda}{J^2}$ . We conjecture that the equivalence extends to all orders in  $\lambda'$  and to higher orders in  $\frac{1}{J}$ .

**KEYWORDS:** AdS-CFT correspondence, Penrose Limit and pp-wave background

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## 1 Introduction

The appearance of integrable structures both at strong and weak coupling has given hope for a complete solution to the spectral problem of the  $\text{AdS}_5/\text{CFT}_4$  duality in the planar limit. This is the best-understood example of a duality between gauge theory and string theory, it states the equivalence between the IIB superstring theory on  $\text{AdS}_5 \times \text{S}^5$  and  $\mathcal{N} = 4$  super Yang-Mills (SYM) theory in  $3 + 1$  dimensions. In [1, 2] an all-loop asymptotic Bethe ansatz has been proposed for the  $\text{AdS}_5/\text{CFT}_4$  duality and starting from the mirror version of the Beisert-Staudacher's equations, further the Y-system was formulated that allows computation of anomalous dimensions for operators of any length [3–6]. This Y-system passes some very important tests: it incorporates the full asymptotic Beisert-Staudacher's Bethe ansatz at large length  $J$  and it reproduces all known wrapping corrections.

Even if the  $\text{AdS}/\text{CFT}$  correspondence is at present best understood for  $\text{AdS}_5$ , also in the more recent  $\text{AdS}_4/\text{CFT}_3$  duality the solution to the spectral problem, at least for the sector described by a coset space, seems to be at reach in the planar limit thanks to integrability [7]. The  $\text{AdS}_4/\text{CFT}_3$  correspondence is an exact duality between type IIA superstring on  $\text{AdS}_4 \times \mathbb{CP}^3$  and a certain regime of the ABJM-theory [8]. The ABJM theory is an Chern-Simons  $\mathcal{N} = 6$  gauge theory with matter dual to M-theory compactified onto  $\text{AdS}_4 \times \text{S}^7/\mathbb{Z}_k$ . It possesses a  $U(N) \times U(N)$  gauge symmetry with Chern-Simons like

kinetic terms at level  $k$  and  $-k$ ; if the 't Hooft coupling  $\lambda = \frac{N}{k}$  is  $1 \ll \lambda \ll k^4$  the gravity side being effectively rendered as a IIA superstring on  $\text{AdS}_4 \times \mathbb{CP}^3$ .

The integrability of the  $\text{AdS}_4/\text{CFT}_3$  duality due to the reduced number of supersymmetries, offers interesting new challenges. The sector of the theory described by a coset space was proven to be classically integrable, but classical integrability for the whole theory has still to be demonstrated [9–13]. Nevertheless, the semiclassical and quantum integrability of some sectors of the theory have received plenty of attention both at weak [14–23] and at strong coupling [24–30]. In particular an all-loop asymptotic Bethe Ansatz has been proposed [7] and a Y-system has been suggested also for the  $\text{AdS}_4/\text{CFT}_3$  duality [3, 4, 6, 31, 73]. More recently nontrivial evidence for the scattering amplitudes/Wilson loop duality for this theory has been given [32–35].

According to the AdS/CFT correspondence energies of excited states of superstrings in specific curved backgrounds should coincide with the anomalous dimensions of appropriate operators of the corresponding gauge field theory. For the planar limit the coupling is zero, however the string is still in a curved space and thus its two-dimensional world-sheet theory is not interaction-free. Calculating the superstring spectrum in such backgrounds is therefore still a complicated problem. Nevertheless the nontrivial interactions become small when one takes the Penrose limit of the metric [36, 37]. Corrections to the free spectrum can then be computed perturbatively as an expansion in inverse powers of the background curvature radius  $R$ .

This idea was suggested by Callan et al. in [38, 39] for  $\text{AdS}_5/\text{CFT}_4$  correspondence. The outcome of [38, 39] and the respective studies on the field theory side [40], have been important for understanding the integrability of the AdS/CFT correspondence. In [38, 39] it was shown for the first time that there is a disagreement between field theory operators anomalous dimensions and the respective string energies at three loops. The disagreement was afterwards interpreted as a breakdown of a double scaling limit and resolved by including the dressing factor that interpolates nontrivially from weak to strong coupling in the Bethe equations describing the spectra of the gauge and the string theory [2, 41–43].

In [44] a complete calculation of the curvature corrections to the pp-wave energy of the two oscillator non-degenerate bosonic states in the decoupled  $\text{SU}(2) \times \text{SU}(2)$  sector of type IIA superstring on  $\text{AdS}_4 \times \mathbb{CP}^3$  was performed. This study was initiated in [26] and reexamined in [28, 45]. In [46] the interacting Hamiltonian for oscillations in the near plane wave limit of  $\text{AdS}_4 \times \mathbb{CP}^3$  was calculated. This is a crucial tool for the computations of this Paper and it is given in [44] by a perturbative expansion in terms of  $1/R$  powers

$$H = H_{2,B} + H_{2,F} + \frac{1}{R} (H_{3,B} + H_{3,BF}) + \frac{1}{R^2} (H_{4,B} + H_{4,F} + H_{4,BF}) + \dots \quad (1.1)$$

where  $R$  is the  $\mathbb{CP}^3$  radius. For brevity we shall further refer to “third-order Hamiltonian”  $H_3 \equiv H_{3,B} + H_{3,BF}$  and “fourth-order Hamiltonian”  $H_4 = H_{4,B} + H_{4,F} + H_{4,BF}$ .

The quadratic Hamiltonian term,  $H_{2B} + H_{2F}$ , is the plane-wave free Hamiltonian from [44] where fermionic and bosonic fields are fully decoupled [15, 24, 47, 48]. A peculiarity of this theory is that in the pp-wave limit the eight massive bosons and eight massive fermions have different worldsheet masses. Four fermions and four bosons are

“heavy”, while the remaining four fermions and four bosons are “light” having a world sheet mass which is 1/2 of that of the heavy ones.

The unique feature of the  $AdS_4$  case is the presence of cubic terms  $H_3$  in the Hamiltonian [26]. This yields extra terms for the matrix element of some arbitrary  $|f\rangle$  state in addition to the expectation value of the quartic Hamiltonian  $H_4$ ,  $\langle f|H_4|f\rangle$ : now the energy correction  $\delta E_f^{(2)}$  looks like

$$\delta E_f^{(2)} = \frac{1}{R^2} \left( \sum_{|i\rangle} \frac{|\langle i|H_3|f\rangle|^2}{E_f - E_{|i\rangle}} + \langle f|H_4|f\rangle \right) \quad (1.2)$$

where  $|i\rangle$  is an intermediate state and summation is done in all admissible channels. The first term in (1.2) gives rise to extra logarithmic divergences. However the total answer must be finite. This result can be achieved by imposing a unique normal ordering prescription as in [44], the ordering prescription being the Weyl prescription.

For readers’ convenience, let us briefly describe here the main characters of this work, referring to [44] for exact definitions. There are four light bosonic oscillators  $a^1, a^2, \tilde{a}^1, \tilde{a}^2$ ; four heavy bosonic oscillators  $\hat{a}^i, i = 1 \dots 4$ ; light fermions  $d_\alpha$ , heavy fermions  $b_\alpha$  where  $\alpha$  is the Dirac ten-dimensional index. The dispersion laws are summarized in the Table 1.

**Table 1.** Dispersion laws

State	Energy
$a^1, a^2$	$\omega_n - c/2$
$\tilde{a}^1, \tilde{a}^2$	$\omega_n + c/2$
$\hat{a}^i$	$\Omega_n$
$d$	$\omega_n$
$b, \text{ even}$	$\Omega_n - c/2$
$b, \text{ odd}$	$\Omega_n + c/2$

where the frequencies are

$$\begin{aligned} \omega_n &= \sqrt{n^2 + \frac{c^2}{4}}, \\ \Omega_n &= \sqrt{n^2 + c^2}. \end{aligned} \quad (1.3)$$

Referring to the “ $SU(2) \times SU(2)$  sector” we mean states solely consisting of either  $a^1, a^2$  or  $\tilde{a}^1, \tilde{a}^2$ . Parameters of the theory are  $c$ , which is meant to be large

$$c = \frac{4J}{R^2}, \quad (1.4)$$

the curvature radius  $R$

$$R^2 = 4\pi\sqrt{2\lambda} = 4\pi J\sqrt{2\lambda'}, \quad (1.5)$$

and the Frolov-Tseytlin coupling constant

$$\lambda' = \frac{\lambda}{J^2}. \quad (1.6)$$

The main result of this Paper is in fact the extension of the results of [44] to the whole set of degenerate two-oscillator light bosonic states, those states whose energies can be also compared directly to the corresponding solutions of the Bethe equations. These are: 8 states built up by two bosonic oscillators and 16 made of two fermionic excitations. They have degenerate plane-wave energy, thus the procedure for computing the spectrum is straightforward, yet technically much more complicated than that in [44]: one must, as in standard quantum mechanical perturbation theory, diagonalize the mixing matrix of the perturbation, solve the secular equation and find eigenvectors and eigenvalues, which are the finite size corrections. The spectrum of such excitations, which we do not display here for brevity (see the Tables on page 15), can be computed exactly in  $\lambda'$  and then compared with the corresponding solutions of the asymptotic Bethe equations: this involves the analysis of configurations carrying auxiliary roots and thus provides a test of the Bethe program even more stringent than the one carried out in [26], where the only activated roots were the fundamental ones, carrying the physical momentum. The Bethe equations must be solved perturbatively, by a judicious Ansatz for the expansion of the momentum in powers of  $\lambda' = \frac{\lambda}{J^2}$  and  $J$  around the asymptotic free solution, and employing a consistent regularization technique for the configurations with auxiliary rapidities 0 or  $\infty$  [49]. Having solved the Bethe equations for the momenta, one plugs the solution in the dispersion relation and gets the spectrum: we obtained it up to  $\mathcal{O}(\lambda^4)$  but it can be improved with some more computational effort. Actually we consider the  $\mathcal{O}(\lambda^4)$  sufficient, since the dressing phase factor interpolating from weak to strong coupling starts at order  $\lambda^3$  and there is no physical mechanism entering at higher orders other than those already encountered. Thus we consider such a matching a very satisfying test to consider it an all order result.

We defer to the main body of the Paper the detailed discussion of the basis that diagonalizes the string theory perturbation Hamiltonian and the corresponding Bethe Ansatz configurations. To summarize we display here the Tables 2,3 of spectrum identifications. They refer to the Bethe configurations in the language of [7]. The integers  $K_i$  are multiplicities of the  $i$ -th Bethe root. The energies of the identified submultiplets are identical at the order  $1/J$ ; this is our main result, which is derived in the main body of the paper.

**Table 2.** Boson-boson state identification

Multiplicity	Corresponding BA states	Corresponding ST states
	$K_4 \ K_4 \ K_3 \ K_2 \ K_1$	State nr.
2	2 0 1 1 1 <sub>branch 1</sub>	5, 7
4	2 0 1 1 1 <sub>branch 2</sub> 1 1 1 1 1 <sub>branch 1</sub>	2, 3, 6, 8
2	1 1 1 1 1 <sub>branch 2</sub>	1, 4

(1.7)

Yet this is not the end of the story about the spectrum of the two-oscillator bosonic states: Each of the 8 bi-bosonic and 16 bi-fermionic state has a further contribution to the energy which is given by the same infinite sum appearing in the eqs. 1.2 and 1.3 of [44].

**Table 3.** Fermion-fermion spectrum comparison

Multiplicity	Corresponding BA states					Corresponding ST states
	$K_4$	$K_{\bar{4}}$	$K_3$	$K_2$	$K_1$	State nr.
2	2	0	2	2	0	23, 24
8	1	1	2	2	0	9, 10, 17, 18, 19, 20, 21, 22
	2	0	2	1	$0_{\text{branch 1}}$	
	2	0	2	1	$0_{\text{branch 2}}$	
	2	0	2	0	0	
6	1	1	2	1	$0_{\text{branch 1}}$	11, 12, 13, 14, 15, 16
	1	1	2	1	$0_{\text{branch 2}}$	
	1	1	2	0	0	

(1.8)

The computation of such term was one of the results of [44] and it is therefore appropriate to recapitulate its interpretation and inquire whether the further developments carried out in this Paper might shed more light about it.

There are no divergences in the eqs. 1.2 and 1.3 of [44] due to a nontrivial, yet natural, ordering prescription for the quantum operator associated to the classical quartic Hamiltonian, which gives infinite sums in the spectrum cancelling those divergencies arising from the cubic Hamiltonian evaluated at second order in perturbation theory. This scheme applies unchanged for the 24 bosonic states considered in this Paper, thus providing further evidence of the naturalness of such ordering prescription.

In [44] it was shown that the infinite sum of Eqs. 1.2 and 1.3 appears diagonally in the mode numbers for the states having an arbitrary number of light bosonic oscillators. Furthermore, if one considers a single-impurity light bosonic state, without the level matching condition which would otherwise have forced its mode number to be vanishing, the energy of this state displays the same kind of contribution. Its natural interpretation is therefore as a correction to the dispersion law of a single magnon. This exponential one-loop effect must be similar to the Lüscher terms coming from a field theory or Bethe Ansatz calculation. One should be able to directly compute it from the Lüscher formula (see the review [50] and references in it). This effect is yet another example of the exponentially small finite size corrections to the magnon dispersion relation that, for type IIA superstring on  $\text{AdS}_4 \times \mathbb{CP}^3$ , were first computed in the giant magnon limit in [25] (see also [51–54]) and derived from Lüscher’s corrections in [55–60]. Finite-size effects were also calculated for spiky strings in  $\text{AdS}_4 \times \mathbb{CP}^3$  and for giant magnons in the presence of an arbitrary two-form  $B$  field. Alternative methods for dealing with giant magnons on  $\text{AdS}_4 \times \mathbb{CP}^3$  by employing the so-called dressing method were suggested in [61–64].

On these grounds it is quite crucial to inquire whether a state built by a light non level matched fermionic oscillator indeed displays the same kind of contribution to the spectrum: such a computation for a fermionic mode actually also involves the issue of quantum ordering of classical terms quartic in the fermions, which was not addressed in [44] since there, they

were not relevant. The generalization to such terms of the Weyl ordering is remarkably the unique choice leading to a finite spectrum. This confirms the naturalness of our ordering prescription. Even more remarkably, for each light fermionic oscillator of a state having an arbitrary (including just one) number of them, one obtains a contribution which is the same infinite sum of Eqs. 1.2 and 1.3 of [44]. This clearly reinforces its interpretation as a finite size correction to the magnon dispersion relation.

The light-magnon dispersion relation is fixed by symmetries of the theory

$$E = \sqrt{\frac{1}{4} + 4h^2(\lambda) \sin^2 \frac{p}{2}} \quad (1.9)$$

but the scaling function  $h(\lambda)$  [15, 24, 47] that interpolates from the strong to the weak coupling is not. The magnon dispersion relation [65, 66] in the AdS<sub>5</sub>/CFT<sub>4</sub> duality is

$$E = \sqrt{1 + f(\lambda) \sin^2 \frac{p}{2}}, \quad (1.10)$$

where  $f(\lambda)$  happens to be equal to  $\frac{\lambda}{\pi^2}$  at both strong and weak coupling. For the AdS<sub>4</sub>/CFT<sub>3</sub> duality the function  $h(\lambda)$  looks like  $\lambda + \mathcal{O}(\lambda^4)$  at weak coupling [14, 15, 47] and like  $\sqrt{\frac{\lambda}{2}} + \mathcal{O}(\lambda^0)$  at strong coupling [15, 25, 47]. It has been computed up to 4 loops on the field theory side in [21, 22, 67]. Quasiclassical calculations for the spinning and folded strings have yielded [27, 58, 68–70]

$$h(\lambda) = \sqrt{\frac{\lambda}{2}} + a_1^{\text{WS}} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right) \quad \text{where} \quad a_1^{\text{WS}} = -\frac{\log 2}{2\pi}, \quad \lambda \gg 1 \quad (1.11)$$

the superscript WS standing for the world-sheet.

Gromov and Vieira, on the other hand by means of the semiclassical Bethe Ansatz [7, 71], extrapolating to the strong coupling of the all loop Ansatz of [67], obtained

$$h(\lambda) = \sqrt{\frac{\lambda}{2}} + a_1^{\text{AC}} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right) \quad \text{where} \quad a_1^{\text{AC}} = 0, \quad \lambda \gg 1 \quad (1.12)$$

where the superscript AC means the algebraic curve.

The different values for  $h(\lambda)$  come from different regularizations used. If one treats all modes in a uniform way, one gets  $a_1 \neq 0$ . If one takes care of heavy and light modes differently, and remembers that heavy modes are kind-of bound states [29] of the light modes and therefore must be cut off at a twice higher value of the momentum as the light ones, one gets the zero  $a_1$ . In this work we argue that there is a definitive evidence from the unitarity preservation requirement to choose a unique regularization, the one with different cutoffs.

Namely, since a “heavy-light-light” vertex is present in the S-matrix, the same cutoff on mode numbers of light and heavy states will render the regularized S-matrix non-unitary. Only cutting the self-energy summation off in such way that preserves unitarity at each large but finite value of the cutoff is acceptable. For conventional global symmetries we know that a regularization breaking a symmetry of a theory results in an anomaly. Unitarity

is a different kind of symmetry, realized on quantum level solely, and not at the level of the classical Lagrangian. However, there is a great degree of resemblance between the  $\log 2$  pieces in the self-energy sums due to broken unitarity by regularization and the presence of the anomalous non-zero parts in otherwise classically zero divergences of Noether currents at quantum level.

Actually the curvature corrections to the string state energies that have been computed in [44] and in this Paper would notice the presence of an  $a_1$  term in  $h(\lambda)$ . In the BMN limit the momentum is  $p = \frac{2\pi n}{J}$  and expanding for large  $J$  at nonzero  $a_1$  in  $h(\lambda)$  yields

$$\begin{aligned}
E &= \sqrt{\frac{1}{4} + 4 \left( \sqrt{\frac{\lambda}{2}} + a_1 \right)^2 \sin^2 \frac{p}{2}} \\
&\simeq \sqrt{\frac{1}{4} + 2\lambda' n^2 \pi^2} + \frac{\sqrt{2} a_1}{J} \left[ 4\pi^2 n^2 \sqrt{\lambda'} - 16\pi^4 n^4 \lambda'^{3/2} + 96\pi^6 n^6 \lambda'^{5/2} + \mathcal{O}(\lambda'^{7/2}) \right]
\end{aligned}
\tag{1.13}$$

namely there will be  $\frac{1}{J} = \frac{4}{cR^2}$  term with semi-integer powers of  $\lambda'$ . Such a term could arise in the finite size energies of two light magnons (see eqs. 1.2 and 1.3 of [44]) however, with the regularization we use, it does not. The  $\lambda'$  power expansion of the first terms in the eqs. 1.2 and 1.3 of [44] yields integer powers of  $\lambda'$ , which are basically due to the interactions between magnons, while the Bessel function sum with produces non analytic terms which are exponentially suppressed like  $\sim e^{-\frac{J}{\sqrt{\lambda}}}$ , thus they are compatible with  $a_1 = 0$  solely.

The Paper is organized as follows. In Section (2) we discuss the dispersion relations of single oscillator states with lifted level matching condition, the regularization procedure and the unitarity argument that proves its consistency. In Section 3 we explicitly compute the energies of the string states and construct the mixing matrix for two-oscillator light bosonic states. In Section 4 we derive the solutions of the Bethe equations corresponding to the string states discussed in Section 3. In Section 5 we draw our conclusions.

## 2 Near plane-wave limit of strings in $\text{AdS}_4 \times \mathbb{CP}^3$ : the procedure

The type IIA superstring of interest to us lives in the  $\text{AdS}_4 \times \mathbb{CP}^3$  background where a two-form and four-form Ramond-Ramond fluxes are present. The corresponding geometry is described in the appendix (A) of [44]. We compute the corrections at the order  $1/R^2 = 1/(4\pi J \sqrt{2\lambda'})$  to the energies of the one- and two particle sectors. We keep  $J$  large,  $\lambda$  large and  $\lambda' = \frac{\lambda}{J^2} = \text{fixed}$ . Thus we can say we are in the near-BMN limit. Our corrections are perturbative quantum mechanical corrections. Unlike the pure BMN case, we already have some interaction in the system. Unlike the folded or rotating string case, semiclassics are not applicable here, so exact quantum-mechanical analysis is due. Unlike the giant magnon case, finite size effects do not decouple from the quantum effects. Thus our limit is in some way unique since it resides at strong coupling, yet is perturbatively treatable.

Our basic string configuration is a point-like IIA string moving in the  $\text{SU}(2) \times \text{SU}(2)$  subsector of  $\mathbb{CP}^3$  and along the time direction  $\mathbb{R}_t$  on  $\text{AdS}_4$  [15, 24, 47]. The specific plane-wave background, has been obtained in [24] and discussed extensively in [26, 46], therefore



we describe it only in the appendix (B) of [44]. The quantization procedure for the free plane-wave Hamiltonian that has been done in the Section 2.1. of [44]. The derivation of the interaction Hamiltonian is found in the Appendix (D) of [44]. All geometry, Gamma matrices, and quantization notations are similar to those of [44]; with the exception of the  $H_{4F}$ , all other Hamiltonian pieces are taken directly from there.

## 2.1 Finite size dispersion relation of single oscillator states and unitarity-preserving Regularization

### Single $u_4$ impurity

We start with heavy bosonic states. Consider the single impurity state, non level matched

$$|u_4\rangle = (a_n^{u_4})^\dagger |0\rangle \quad (2.1)$$

Cancellation of divergencies for bosons is a crucial test for the validity of our theory. To illustrate how divergencies cancel in the dispersion relation for the fourth heavy boson, we show the partial contributions of different sectors in the Table (4) below. The boson energy is

$$E_n^{u_4} = \Omega_n + \frac{1}{R^2 \Omega_n} \sum_q \epsilon_q^{u_4}. \quad (2.2)$$

**Table 4.** Cancellation of divergencies for bosons

Hamiltonian piece	$\epsilon_q^{u_4}$
$H_3^2$	$-\frac{4q^2}{c\omega_q} + \frac{n^2}{\omega_q\omega_{n+q}} + \frac{2nq}{\omega_q\omega_{n+q}}$
$H_{\hat{u}_4\hat{u}_4BB}^{light}$	$-\frac{8n^2q^2}{c^3\omega_q} - \frac{8nq\Omega_n}{c^3} - \frac{2n^2}{c\omega_q} - \frac{4q^2}{c\omega_q}$
$H_{\hat{u}_4\hat{u}_4BB}^{heavy}$	$\frac{6nq\Omega_n}{c^3} - \frac{6q^2\Omega_n^2}{c^3\Omega_q}$
$H_{4\hat{u}_4}$	$-\frac{2n^2q^2}{c^3\Omega_q} + \frac{2nq\Omega_n}{c^3} - \frac{2n^2}{c\Omega_q} - \frac{2q^2}{c\Omega_q}$
$H_{2B2F}^{light}$	$\frac{8q^2\Omega_n^2}{c^3\omega_q} - \frac{8nq\Omega_n}{c^2}$
$H_{2B2F}^{heavy}$	$\frac{8q^2\Omega_n^2}{c^3\Omega_q} - \frac{8nq\Omega_n}{c^2} + \frac{4n^2}{c\Omega_q}$
Total	$\frac{2n^2}{c} \left( \frac{1}{\Omega_q} - \frac{1}{\omega_q} \right)$

The “total” line of the table refers to the sum both over partial contributions and the summation mode index. This lets the expressions be additionally simplified, since the dumb variable allows constant shifts, leading to extra cancellations. Also note exact cancellation of quadratic divergencies. By summing over partial channels shown above the dispersion relation up to finite size becomes

$$E^{u_4} = \sqrt{1 + \frac{n^2}{c^2}} + \frac{2n^2}{c\Omega_n R^2} \left( \sum_{q=-2N}^{2N} \frac{1}{\Omega_q} - \sum_{q=-N}^N \frac{1}{\omega_q} \right) +$$

$$\begin{aligned} & \frac{1}{cR^2\Omega_n} \sum_{q=-N}^N \left\{ 4\frac{q^2}{\omega_q} - (2\omega_q + \omega_{q+n} + \omega_{q-n}) + \frac{c^2}{4} \left( \frac{2}{\omega_q} + \frac{1}{\omega_{q+n}} + \frac{1}{\omega_{q-n}} \right) \right. \\ & \left. - \frac{c}{2} \left( \frac{\omega_{n+q}}{\omega_q} - \frac{\omega_q}{\omega_{n+q}} + \frac{\omega_{-n+q}}{\omega_q} - \frac{\omega_q}{\omega_{-n+q}} \right) \right\} \end{aligned} \quad (2.3)$$

We already know very well how to treat the sum in the first line in eq. (2.3) since it is exactly the one appearing in [44], giving rise to the Bessel functions series. For this state, this sum appears uniquely from contributions due to the quartic Hamiltonian. The other terms in the equation are organized as follows: their cutoff is  $N$  because the sum is over a light mode.

Since the sum in the second and third lines of (2.3) are convergent and have the same cutoff  $N$ , we can safely send it to infinity and performing some shifts we can see that all the terms sum up to zero. Therefore the dispersion relation is rather the following simpler one:

$$E^{u_4} = \sqrt{1 + \frac{n^2}{c^2}} + \frac{2n^2}{c\Omega_n R^2} \left( \sum_{q=-2N}^{2N} \frac{1}{\Omega_q} - \sum_{q=-N}^N \frac{1}{\omega_q} \right) \quad (2.4)$$

### Single $u_1$ impurity

The interaction Hamiltonian is not explicitly invariant with regard to  $u_4 \rightarrow u_i$  replacement, since the fourth direction is special it belongs to  $\mathbb{CP}^3$  while  $u_i \in \text{AdS}_4$  for  $i = 1, 2, 3$ . Therefore we must consider now the single impurity heavy state with a  $u_1$  oscillator separately

$$|u_1\rangle = (a_n^{u_1})^\dagger |0\rangle \quad (2.5)$$

We see by an explicit calculation that its dispersion relation up to finite size is the same as for the  $u_4$  single oscillator state:

$$E^{u_1} = \sqrt{1 + \frac{n^2}{c^2}} + \frac{2n^2}{c\Omega_n R^2} \left( \sum_{q=-2N}^{2N} \frac{1}{\Omega_q} - \sum_{q=-N}^N \frac{1}{\omega_q} \right) \quad (2.6)$$

By virtue of the same argument as above we can see that all divergencies cancel, whereas the remaining term contains only an exponentially small correction in  $J$ .

### Fermions

Consider now the fermionic states: the light one

$$|d\rangle = d_{\alpha n}^\dagger |0\rangle, \quad (2.7)$$

and the heavy one

$$|b\rangle = b_{\alpha n}^\dagger |0\rangle. \quad (2.8)$$

We check the fermion dispersion relation perturbatively and demonstrate the results in the Table (5).

**Table 5.** Cancellation of divergencies for fermions. Separate sectors give divergent results, the remnant is finite.

Hamiltonian piece	Light state energy correction $\epsilon^d$	Heavy state state energy correction $\epsilon^b$
$H_3^2$ light loop	$-\frac{3n^2}{16c\omega_q} - \frac{3q^2}{16c\omega_q} + \frac{3q^2}{16c\Omega_q}$	$-\frac{n^2}{8c\omega_q}$
$H_3^2$ heavy loop	$-\frac{n^2}{4c\Omega_q} + \frac{q^2}{16c\omega_q} - \frac{9c}{64\omega_q} - \frac{q^2}{16c\Omega_q} - \frac{9c}{32\Omega_q}$	0
$H_{2B2F}$ light loop	$-\frac{2q^2n^2}{c^3\omega_q} - \frac{n^2}{2c\omega_q} - \frac{q^2}{16c\omega_q} + \frac{9c}{64\omega_q}$	$-\frac{4q^2n^2}{c^3\omega_q} - \frac{n^2}{c\omega_q} - \frac{5q^2}{2c\omega_q} - \frac{9c}{16\omega_q}$
$H_{2B2F}$ heavy loop	$-\frac{2q^2n^2}{c^3\Omega_q} - \frac{n^2}{2c\Omega_q} - \frac{3q^2}{16c\Omega_q}$	$-\frac{4q^2n^2}{c^3\Omega_q} - \frac{n^2}{c\Omega_q} - \frac{2q^2}{c\Omega_q}$
$H_{4F}$ light loop	$\frac{2q^2n^2}{c^3\omega_q} + \frac{5n^2}{16c\omega_q} + \frac{5q^2}{16c\omega_q}$	$\frac{4q^2n^2}{c^3\omega_q} + \frac{n^2}{8c\omega_q} + \frac{5q^2}{2c\omega_q} + \frac{9c}{16\omega_q}$
$H_{4F}$ heavy loop	$-\frac{n^2}{8c\omega_q} + \frac{2q^2n^2}{c^3\Omega_q} + \frac{5n^2}{4c\Omega_q} - \frac{q^2}{8c\omega_q} + \frac{q^2}{16c\Omega_q} + \frac{9c}{32\Omega_q}$	$\frac{4q^2n^2}{c^3\Omega_q} + \frac{2n^2}{c\Omega_q} + \frac{2q^2}{c\Omega_q}$
Total	$\frac{n^2}{2c\Omega_q} - \frac{n^2}{2c\omega_q}$	$\frac{n^2}{c\Omega_q} - \frac{n^2}{c\omega_q}$

The superficial divergences present in the loop contributions to fermions do cancel indeed, only a finite exponentially suppressed part (as  $e^{-constJ}$ ) remaining. Here we demonstrate how various contributions cancel in order to leave a finite piece only. For light states

$$E^d = \omega_n + \frac{1}{R^2\omega_n} \sum_q \epsilon_q^d, \quad (2.9)$$

for heavy states

$$E^b = \Omega_n + \frac{1}{R^2\Omega_n} \sum_q \epsilon_q^b. \quad (2.10)$$

The partial (taken over separate channels)  $\epsilon_q^d$  and  $\epsilon_q^b$  are given in the Table (5).

### Unitarity-preserving Regularization

The sums of the light and heavy self-energies  $\sum_q \epsilon_q^d$ ,  $\sum_q \epsilon_q^b$  are convergent, but have to be regularized to be ascribed a numerical value. We use here the most natural “algebraic-curve” regularization prescription suggested by the form of the cubic Hamiltonian [44]. That is, we cut the heavy modes at a cutoff  $2N$ , and the light modes at a cutoff  $N$ , where  $N$  is afterwards sent to infinity. The arguments that have been used in literature for this regularization have been reiterated below and will be discussed yet once more in the Conclusion; here we wish to bring in a very generic argument, which leaves this “unequal-frequency” regularization as the only permissible one. Zarembo has shown [29] that the heavy-to-light vertex is organized in our theory in such a way that the light two-particle cut starts exactly in the point where the heavy particle pole is. Thus an on-shell decay heavy-into-two-light modes is seen to be possible. Consider now the most general requirement of validity of a quantum field theory, the equation on the unitarity of the S-matrix

$$SS^\dagger = 1 \quad (2.11)$$

Let us write it down more explicitly in the 1-particle heavy sector (indices  $1, i$ ) and the 2-particle sector (indices  $2, jk$ )

$$S_{im}^{1,1} S_{mi'}^{1,1\dagger} + S_{i,jk}^{1,2} S_{jk,i'}^{1,2\dagger} = 1_{i,i'}^{1,1} \quad (2.12)$$

where summation is meant over the repeating Hilbert space indices. The second part of the left-hand side must be taken into account due to the mentioned result by Zarembo. The relation is valid at any mode number. Suppose we regularize the theory now. This effectively means that both for the unit operator in the Hilbert space and for the S-matrices all elements above some  $N_{cutoff}$  are filled in with zeros. Suppose that  $N_{cutoff}$  is special for each of the sectors. Thus we have below the cutoffs

$$S_{2n,2n}^{1,1} S_{2n,2n}^{1,1\dagger} \Big|_{2n < N_{cutoff}^{heavy}} + S_{2n,n}^{1,2} S_{n,n,2n}^{1,2\dagger} \Big|_{n < N_{cutoff}^{light}} = 1_{2n,2n}^{1,1} \Big|_{2n < N_{cutoff}^{heavy}}, \quad (2.13)$$

$$S_{2n,2n}^{1,1} S_{2n,2n}^{1,1\dagger} \Big|_{2n > N_{cutoff}^{heavy}} + S_{2n,n}^{1,2} S_{n,n,2n}^{1,2\dagger} \Big|_{n > N_{cutoff}^{light}} = 0. \quad (2.14)$$

Then we can see that the only way to comply with unitarity is to impose  $N_{cutoff}^{heavy} = 2N_{cutoff}^{light}$ . The intuitive way to understand this physics is very simple: if you cut the Hilbert space off at an arbitrary energy, the heavy modes will decay into nothing. This is precisely what we usually understand as a non-unitary theory - a non-unity-normalized total probability of an inclusive process (in our case, it's "heavy mode into something").

Thus we impose the regularization and calculate the sums above as was done in [44]. We obtain

$$\delta E \sim e^{-J/\sqrt{2\lambda}}, \quad (2.15)$$

which means there are no power corrections to the energy. From these results we get in the strong-coupling limit the function  $h(\lambda)$ , parameterized as

$$h(\lambda) = \sqrt{\frac{\lambda}{2}} + a_1 + \dots, \quad (2.16)$$

that

$$a_1 = 0, \quad (2.17)$$

supporting the result from the Bethe Ansatz algebraic curve.

### 3 Finite-size mixing matrix for two-oscillator light bosonic states

The mixing matrix between two-oscillator bosonic states having degenerate plane-wave energies is

$$M_{\text{mix}}^{ij} = \sum_{|i\rangle} \frac{\langle e_i | H_{(3)} | i \rangle \langle i | H_{(3)} | e_j \rangle}{E_{|e_i\rangle}^{(0)} - E_{|i\rangle}^{(0)}} + \langle e_i | H_4 | e_j \rangle \quad (3.1)$$

Solving the secular equation for  $M_{\text{mix}}^{ij}$  one gets the eigenvectors and the eigenvalues, i.e. the finite size corrections to the spectrum. The four single-oscillator light bosonic states are

$$\{a_n^{1\dagger}|0\rangle, \tilde{a}_n^{1\dagger}|0\rangle, a_n^{2\dagger}|0\rangle, \tilde{a}_n^{2\dagger}|0\rangle\}, \quad (3.2)$$

and the single-oscillator light fermionic states are  $d_{\alpha,n}^\dagger|0\rangle$ .

We can build eight degenerate states with two bosonic oscillators and sixteen physical states with two fermionic oscillators, all of them having plane-wave energy  $2\omega_n$ , in units of  $c$ . There are other eight bibosonic states, which are non-degenerate and have been considered in [44]. An educated guess, on the grounds of the symmetries of the Hamiltonian, on the choice of the basis  $|e_i\rangle$  shall sharpen the computation considerably and that's what we are up to. One could naively build these four bosonic states:

$$\begin{aligned} v_n^1 &= a_n^{1\dagger} \tilde{a}_{-n}^{1\dagger} |0\rangle, \\ v_n^2 &= a_n^{1\dagger} \tilde{a}_{-n}^{2\dagger} |0\rangle, \\ v_n^3 &= a_n^{2\dagger} \tilde{a}_{-n}^{1\dagger} |0\rangle, \\ v_n^4 &= a_n^{2\dagger} \tilde{a}_{-n}^{2\dagger} |0\rangle. \end{aligned} \quad (3.3)$$

where  $n > 0$ , and equally states with  $n \rightarrow -n$ . The true basis should however possess definite parities with respect to  $\mathbb{Z}_2$  symmetries: the momentum reflection symmetry  $P_n : n \rightarrow -n$ , the symmetry between the two  $SU(2)$ 's  $P_a : a^1 \rightarrow a^2$ , and the symmetry  $\tilde{P} : a_i \rightarrow \tilde{a}_i$ . Such states are easily constructed as follows: first symmetrize and antisymmetrize in  $P_a$

$$\begin{aligned} s_n &= \frac{1}{\sqrt{2}}(v_n^1 + v_n^4) \\ p_n &= \frac{1}{\sqrt{2}}(-v_n^1 + v_n^4) \\ q_n &= \frac{1}{\sqrt{2}}(v_n^2 + v_n^3) \\ r_n &= \frac{1}{\sqrt{2}}(-v_n^2 + v_n^3). \end{aligned} \quad (3.4)$$

The full basis of Lorenzian spin zero light two-oscillator boson-boson tree-level degenerate states has then dimension eight and is, after decomposition into  $P_n$  even and odd states:

state	definition	$P_n$	$\tilde{P}$	$P_a$
$u_1$	$\frac{1}{\sqrt{2}}(s_n + s_{-n})$	1	1	1
$u_2$	$\frac{1}{\sqrt{2}}(-s_n + s_{-n})$	-1	-1	1
$u_3$	$\frac{1}{\sqrt{2}}(p_n + p_{-n})$	1	1	-1
$u_4$	$\frac{1}{\sqrt{2}}(-p_n + p_{-n})$	-1	-1	-1
$u_5$	$\frac{1}{\sqrt{2}}(q_n + q_{-n})$	1	1	1
$u_6$	$\frac{1}{\sqrt{2}}(-q_n + q_{-n})$	-1	-1	1
$u_7$	$\frac{1}{\sqrt{2}}(r_n + r_{-n})$	1	-1	-1
$u_8$	$\frac{1}{\sqrt{2}}(-r_n + r_{-n})$	-1	1	-1

(3.5)

The two-fermion-oscillator states are of the type  $d_{\alpha}^\dagger A_{\alpha\beta} d_{\beta}^\dagger |0\rangle$ , where the ones out of them with zero AdS spin  $s$  can potentially mix with the light bosonic states as well.  $A_{\alpha\beta}$  is an

arbitrary matrix with fermionic indices. Choosing the linearly independent states by the following projection criteria:

$$\begin{aligned}
\Gamma_{11}^T A \Gamma_{11} &\neq 0, \\
\Gamma_+^T A \Gamma_+ &\neq 0, \\
\mathcal{P}^T A \mathcal{P} &\neq 0
\end{aligned} \tag{3.6}$$

we find the fermionic-fermionic basis has dimension sixteen, as physically expected since the light physical degrees of freedom of each 32-dimensional spinor of our theory are four. Therefore, the total basis in the mixing sector has 24 dimensions; the states  $u_{9..24} = \{d^+ A_i d^+ | 0\rangle\}$  can be chosen as

$$\begin{aligned}
A_{i=9..24} = & \left\{ \frac{1}{2}, \frac{1}{2}\Gamma_{56}, \frac{1}{2}\Gamma_{12}, \frac{1}{2}\Gamma_{13}, \frac{1}{2}\Gamma_{23}, \right. \\
& \frac{1}{2}\Gamma_{1256}, \frac{1}{2}\Gamma_{1356}, \frac{1}{2}\Gamma_{2356}, \frac{1}{2}\Gamma_{1457}, \frac{1}{2}\Gamma_{2457}, \\
& \left. \frac{1}{2}\Gamma_{3457}, \frac{1}{2}\Gamma_{1467}, \frac{1}{2}\Gamma_{2467}, \frac{1}{2}\Gamma_{3467}, \frac{1}{2}\Gamma_{0579}, \frac{1}{2}\Gamma_{0679} \right\}.
\end{aligned} \tag{3.7}$$

where the index  $i$  numbering basis states runs from 9 to 24. It can be explicitly seen that this basis is orthonormal.

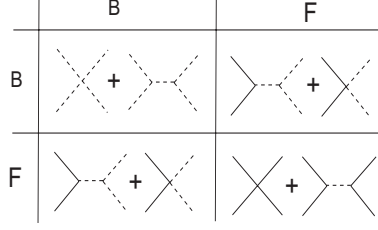
The computation of the mixing matrix is standard. When taking the matrix elements, we consider only the contraction combinations in which the oscillators of the quartic Hamiltonian are all contracted with those of the external states. For the cubic Hamiltonian evaluated at second order in perturbation theory, we keep only the combinations of contractions which lead to enough delta's on the mode numbers as to determine all the free indices of the intermediate states as functions of the external state ones,  $n$  and  $-n$ . We are then discarding all the contractions bringing contributions to the spectrum as infinite sums over a free summand: these have actually been addressed separately and they lead to the finite-size correction to the dispersion relation, which is a peculiarity of this theory, as discussed in the Introduction and in [44].

Schematically, we may picture the mixing matrix as follows

$$\begin{array}{c|cc}
& B & F \\
\hline
B & H_{4B} + H_{3B}^2 & H_{2B2F} + H_{FFB}H_{BBF} \\
F & H_{2B2F} + H_{FFB}H_{BBF} & H_{4F} + H_{FB}^2
\end{array} \tag{3.8}$$

where  $F$  and  $B$  represent the two-fermion and two-boson oscillator states. This Hamiltonian is symbolically depicted in Fig. (1).

The corresponding pieces of the mixing matrix are shown below.



**Figure 1.** Schematic diagram of the contributing sectors of string Hamiltonian

The four-boson contribution to the BB sector is the following  $8 \times 8$  matrix given in terms of the basis states  $u_1 \dots u_8$ :

$$H_{4B} = \frac{2n^2}{c\omega_n^2} \text{Diagonal} \left( -\frac{4n^2}{c^2} - 4, -\frac{4n^2}{c^2} - 3, -\frac{4n^2}{c^2} - 4, -\frac{4n^2}{c^2} - 5, -\frac{4n^2}{c^2}, -\frac{4n^2}{c^2} - 1, -\frac{4n^2}{c^2}, -\frac{4n^2}{c^2} - 1 \right). \quad (3.9)$$

The  $H_{3B}$  contributes via three types of intermediate states

$$H_{3B}^2 = \sum_{j=1}^3 H_{3B_j}, \quad (3.10)$$

where the intermediate states  $s_j$  are:

$$\begin{aligned} |s_1\rangle &= |\hat{a}_0^{4\dagger}|0\rangle, \\ |s_2\rangle &= |\hat{a}_{-a-b}^{4\dagger} a_a^\dagger \tilde{a}_b^\dagger|0\rangle, \\ |s_3\rangle &= |\hat{a}_{-a-b-c-d}^{4\dagger} a_a^\dagger \tilde{a}_b^\dagger a_c^\dagger \tilde{a}_d^\dagger|0\rangle. \end{aligned} \quad (3.11)$$

The following matrix (in terms of the same bosonic basis as above) is the contribution of the first channel

$$H_{3B_1}^2 = \frac{4n^2}{c\omega_n^2} \text{Diagonal} \left( 0, 0, \omega_n + \frac{c}{2}, 0, 0, 0, 0, 0 \right). \quad (3.12)$$

The second channel gives

$$H_{3B_2}^2 = \frac{2n^2}{c\omega_n^2} \text{Diagonal} \left( 0, 1, 0, 1, 0, -1, 0, -1 \right). \quad (3.13)$$

Finally the third channel gives

$$H_{3B_3}^2 = -\frac{4n^2}{c\omega_n^2} \text{Diagonal} \left( 0, 0, \omega_n - \frac{c}{2}, 0, 0, 0, 0, 0 \right). \quad (3.14)$$

Switch now to the two fermionic oscillators. The quartic purely fermionic Hamiltonian is

$$\begin{aligned} \mathcal{H}_{4,F} &= -\frac{i}{24} \left( \bar{\theta} \Gamma_{11} \Gamma_+ \mathcal{M}^2 \theta' + \bar{\theta} \Gamma_+ \mathcal{M}^2 \Gamma_{11} \theta' \right) - \frac{1}{2c} (A_{+, \sigma}^2 - \tilde{A}_{+, \sigma}^2) \\ &\quad - \frac{1}{4} A_{+, \sigma} (\tilde{C}_{+-} + \tilde{B}_{+56} + \tilde{B}_{+78}) + \frac{1}{4} \tilde{A}_{+, \sigma} (C_{+-} - C_{++} + B_{+56} + B_{+78}). \\ &\quad - \frac{c}{8} \sum_{i=1}^4 C_{+i}^2 - \frac{c}{32} \sum_{i=5}^8 \left[ 2C_{+i} - s_i B_{+4i} + \frac{1}{2} \sum_{j=5}^8 \epsilon_{ij} B_{+-j} \right]^2 \end{aligned} \quad (3.15)$$

Unlike the other sectors where we have referred the reader to [44] for the Hamiltonian expression, we show  $\mathcal{H}_{4,F}$  explicitly, since we correct a misprint of the earlier version.

In the basis  $u_9 \dots u_{24}$  the  $16 \times 16$  matrix of the mixing in the fermionic sector is

$$H_{4F} = -\frac{2n^2}{c\omega_n^2} \text{Diagonal} \left( \frac{4n^2}{c^2} + \frac{3}{2}, \frac{4n^2}{c^2} - \frac{3}{2}, \frac{4n^2}{c^2} + \frac{5}{2}, \frac{4n^2}{c^2} + \frac{5}{2}, \frac{4n^2}{c^2} + \frac{5}{2}, \frac{4n^2}{c^2} + \frac{3}{2}, \frac{4n^2}{c^2} + \frac{3}{2}, \frac{4n^2}{c^2} + \frac{3}{2}, \frac{4n^2}{c^2} + \frac{3}{2}, \frac{4n^2}{c^2} + \frac{1}{2}, \frac{4n^2}{c^2} + \frac{1}{2}, \frac{4n^2}{c^2} + \frac{1}{2}, \frac{4n^2}{c^2} + \frac{1}{2}, \frac{4n^2}{c^2} + \frac{1}{2}, \frac{4n^2}{c^2} - \frac{1}{2}, \frac{4n^2}{c^2} - \frac{1}{2} \right). \quad (3.16)$$

The mixing term in the FF sector coming from the  $H_{FFB}^2$  is the following:

$$H_{FFB}^2 = \frac{2n^2}{c\omega_n^2} \text{Diagonal} \left( \frac{3}{2}, -\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2} \right). \quad (3.17)$$

This term comes from the long intermediate channel; the short channel yields identically 0. Of the 16 fermionic states,  $u_9 \dots u_{24}$  only states  $u_9, u_{10}$  have Lorentzian  $AdS$  spin  $s = 0$  and thus could have in principle mixed with bibosonic states, all of which have spin 0. This mixing indeed is vanishing. There could be in principle a mixed term  $H_{BBB}H_{BFF}$ , with a bosonic zero mode intermediate state as shown in Fig. (1), but explicit calculation shows it is zero. Summing the contributions we get the full mixing matrix, which our judicious choice of the basis has automatically brought to a diagonal form. We have therefore obtained the set of eigenstates and eigenvalues exact in  $\lambda'$ . The following is the finite-size spectrum of two bosonic oscillations, where, on the rightmost column, in order for further comparison with the Bethe Ansatz, we have expressed the spectrum in terms of  $\lambda'$  and  $J$ , expanding up to fourth order in  $\lambda'$ : see table 6. The finite-size spectrum of the two fermionic oscillations

**Table 6.** Finite-size spectrum of two bosonic oscillations

state	spectrum	expansion of the spectrum
$u_1$	$-\frac{8n^2\Omega_n^2}{c^3R^2\omega_n^2}$	$\frac{1}{J} (-16n^2\pi^2\lambda' + 96n^4\pi^4\lambda'^2 - 768n^6\pi^6\lambda'^3 + 6144n^8\pi^8\lambda'^4 + \dots)$
$u_2$	$-\frac{4n^2(c^2+2n^2)}{c^3R^2\omega_n^2}$	$\frac{1}{J} (-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + 2048n^8\pi^8\lambda'^4 + \dots)$
$u_3$	$-\frac{4n^2(c^2+2n^2)}{c^3R^2\omega_n^2}$	$\frac{1}{J} (-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + 2048n^8\pi^8\lambda'^4 + \dots)$
$u_4$	$-\frac{8n^2\Omega_n^2}{c^3R^2\omega_n^2}$	$\frac{1}{J} (-16n^2\pi^2\lambda' + 96n^4\pi^4\lambda'^2 - 768n^6\pi^6\lambda'^3 + 6144n^8\pi^8\lambda'^4 + \dots)$
$u_5$	$-\frac{8n^4}{c^3R^2\omega_n^2}$	$\frac{1}{J} (-32n^4\pi^4\lambda'^2 + 256n^6\pi^6\lambda'^3 - 2048n^8\pi^8\lambda'^4 + \dots)$
$u_6$	$-\frac{4n^2(c^2+2n^2)}{c^3R^2\omega_n^2}$	$\frac{1}{J} (-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + 2048n^8\pi^8\lambda'^4 + \dots)$
$u_7$	$-\frac{8n^4}{c^3R^2\omega_n^2}$	$\frac{1}{J} (-32n^4\pi^4\lambda'^2 + 256n^6\pi^6\lambda'^3 - 2048n^8\pi^8\lambda'^4 + \dots)$
$u_8$	$-\frac{4n^2(c^2+2n^2)}{c^3R^2\omega_n^2}$	$\frac{1}{J} (-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + 2048n^8\pi^8\lambda'^4 + \dots)$

is given in the Table 7 below. The Tables above show that each energy has an even, at least double, multiplicity, as one expects from the symmetry of the Bethe framework configurations, as we shall discuss in the Section below. Thus the above string spectrum, one of the main results of this Paper, can be consistently compared with the solutions of



**Table 7.** Finite-size spectrum of two fermionic oscillations

state	spectrum	expansion of the spectrum
$u_9$	$-\frac{8n^4}{c^3 R^2 \omega_n^2}$	$\frac{1}{J} (-32n^4 \pi^4 \lambda'^2 + 256n^6 \pi^6 \lambda'^3 - 2048n^8 \pi^8 \lambda'^4 + \dots)$
$u_{10}$	$-\frac{8n^4}{c^3 R^2 \omega_n^2}$	$\frac{1}{J} (-32n^4 \pi^4 \lambda'^2 + 256n^6 \pi^6 \lambda'^3 - 2048n^8 \pi^8 \lambda'^4 + \dots)$
$u_{11}$	$-\frac{4n^2(c^2+2n^2)}{c^3 R^2 \omega_n^2}$	$\frac{1}{J} (-8n^2 \pi^2 \lambda' + 32n^4 \pi^4 \lambda'^2 - 256n^6 \pi^6 \lambda'^3 + 2048n^8 \pi^8 \lambda'^4 + \dots)$
$u_{12}$	$-\frac{4n^2(c^2+2n^2)}{c^3 R^2 \omega_n^2}$	$\frac{1}{J} (-8n^2 \pi^2 \lambda' + 32n^4 \pi^4 \lambda'^2 - 256n^6 \pi^6 \lambda'^3 + 2048n^8 \pi^8 \lambda'^4 + \dots)$
$u_{13}$	$-\frac{4n^2(c^2+2n^2)}{c^3 R^2 \omega_n^2}$	$\frac{1}{J} (-8n^2 \pi^2 \lambda' + 32n^4 \pi^4 \lambda'^2 - 256n^6 \pi^6 \lambda'^3 + 2048n^8 \pi^8 \lambda'^4 + \dots)$
$u_{14}$	$-\frac{4n^2(c^2+2n^2)}{c^3 R^2 \omega_n^2}$	$\frac{1}{J} (-8n^2 \pi^2 \lambda' + 32n^4 \pi^4 \lambda'^2 - 256n^6 \pi^6 \lambda'^3 + 2048n^8 \pi^8 \lambda'^4 + \dots)$
$u_{15}$	$-\frac{4n^2(c^2+2n^2)}{c^3 R^2 \omega_n^2}$	$\frac{1}{J} (-8n^2 \pi^2 \lambda' + 32n^4 \pi^4 \lambda'^2 - 256n^6 \pi^6 \lambda'^3 + 2048n^8 \pi^8 \lambda'^4 + \dots)$
$u_{16}$	$-\frac{4n^2(c^2+2n^2)}{c^3 R^2 \omega_n^2}$	$\frac{1}{J} (-8n^2 \pi^2 \lambda' + 32n^4 \pi^4 \lambda'^2 - 256n^6 \pi^6 \lambda'^3 + 2048n^8 \pi^8 \lambda'^4 + \dots)$
$u_{17}$	$-\frac{8n^4}{c^3 R^2 \omega_n^2}$	$\frac{1}{J} (-32n^4 \pi^4 \lambda'^2 + 256n^6 \pi^6 \lambda'^3 - 2048n^8 \pi^8 \lambda'^4 + \dots)$
$u_{18}$	$-\frac{8n^4}{c^3 R^2 \omega_n^2}$	$\frac{1}{J} (-32n^4 \pi^4 \lambda'^2 + 256n^6 \pi^6 \lambda'^3 - 2048n^8 \pi^8 \lambda'^4 + \dots)$
$u_{19}$	$-\frac{8n^4}{c^3 R^2 \omega_n^2}$	$\frac{1}{J} (-32n^4 \pi^4 \lambda'^2 + 256n^6 \pi^6 \lambda'^3 - 2048n^8 \pi^8 \lambda'^4 + \dots)$
$u_{20}$	$-\frac{8n^4}{c^3 R^2 \omega_n^2}$	$\frac{1}{J} (-32n^4 \pi^4 \lambda'^2 + 256n^6 \pi^6 \lambda'^3 - 2048n^8 \pi^8 \lambda'^4 + \dots)$
$u_{21}$	$-\frac{8n^4}{c^3 R^2 \omega_n^2}$	$\frac{1}{J} (-32n^4 \pi^4 \lambda'^2 + 256n^6 \pi^6 \lambda'^3 - 2048n^8 \pi^8 \lambda'^4 + \dots)$
$u_{22}$	$-\frac{8n^4}{c^3 R^2 \omega_n^2}$	$\frac{1}{J} (-32n^4 \pi^4 \lambda'^2 + 256n^6 \pi^6 \lambda'^3 - 2048n^8 \pi^8 \lambda'^4 + \dots)$
$u_{23}$	$\frac{4n^2(c^2-2n^2)}{c^3 R^2 \omega_n^2}$	$\frac{1}{J} (8n^2 \pi^2 \lambda' - 96n^4 \pi^4 \lambda'^2 + 768n^6 \pi^6 \lambda'^3 - 6144n^8 \pi^8 \lambda'^4 + \dots)$
$u_{24}$	$\frac{4n^2(c^2-2n^2)}{c^3 R^2 \omega_n^2}$	$\frac{1}{J} (8n^2 \pi^2 \lambda' - 96n^4 \pi^4 \lambda'^2 + 768n^6 \pi^6 \lambda'^3 - 6144n^8 \pi^8 \lambda'^4 + \dots)$

the Bethe equations, providing a significant test of them and of the integrability of the string sigma model in the near-BMN limit. Notice also that the results above, together with the eight bosonic states addressed in [44], sheds a complete light on the spectrum of all the 32 bosonic two-impurity light states of the theory. Finally, for comparison with the Bethe Ansatz framework, the string spectrum above, exact in  $\lambda'$ , can be expanded in power series, as shown in the Table 8 below. The  $c_i$  are the coefficients of the  $\lambda'$  expansion of the spectrum defined as

$$\epsilon = \epsilon_0 + \frac{1}{J} \sum_{i=1} c_i \lambda'^i (8\pi^2 n^2)^i + \mathcal{O}\left(\frac{1}{J^2}\right), \quad (3.18)$$

where  $\epsilon_0$  is the term of order  $1/J^0$ ,  $n$  is the mode number of the two-oscillator state.

**Table 8.** Coefficients  $c_i$  and state multiplicities.

State type (BB,FF) and Nr. $i$	$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	Multiplicity
$FF : 23, 24$	1	$-\frac{3}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$	$2_{FF}$
$BB : 5, 7; FF : 9, 10, 17, 18, 19, 20, 21, 22$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$10 = 8_{FF} + 2_{BB}$
$BB : 2, 3, 6, 8; FF : 11, 12, 13, 14, 15, 16$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$10 = 6_{FF} + 4_{BB}$
$BB : 1, 4$	-2	$\frac{3}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	$2_{BB}$

(3.19)

#### 4 Energies of Bethe states

The Bethe roots are quantized through the algebraic equations [7]:

$$\begin{aligned}
1 &= \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + \frac{i}{2}}{u_{1,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{1 - 1/x_{1,k}x_{4,j}^+}{1 - 1/x_{1,k}x_{4,j}^-} \prod_{j=1}^{K_{\bar{4}}} \frac{1 - 1/x_{1,k}x_{\bar{4},j}^+}{1 - 1/x_{1,k}x_{\bar{4},j}^-}, \\
1 &= \prod_{j \neq k}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + \frac{i}{2}}{u_{2,k} - u_{1,j} - \frac{i}{2}} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}}{u_{2,k} - u_{3,j} - \frac{i}{2}}, \\
1 &= \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}}{u_{3,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k} - x_{4,j}^+}{x_{3,k} - x_{4,j}^-} \prod_{j=1}^{K_{\bar{4}}} \frac{x_{3,k} - x_{\bar{4},j}^+}{x_{3,k} - x_{\bar{4},j}^-}, \\
\left( \frac{x_{4,k}^+}{x_{4,k}^-} \right)^L &= \prod_{j \neq k}^{K_4} \frac{u_{4,k} - u_{4,j} + i}{u_{4,k} - u_{4,j} - i} \prod_{j=1}^{K_1} \frac{1 - 1/x_{4,k}^-x_{1,j}}{1 - 1/x_{4,k}^+x_{1,j}} \prod_{j=1}^{K_3} \frac{x_{4,k}^- - x_{3,j}}{x_{4,k}^+ - x_{3,j}} \times \\
&\times \prod_{j=1}^{K_4} \sigma_{\text{BES}}(u_{4,k}, u_{4,j}) \prod_{j=1}^{K_{\bar{4}}} \sigma_{\text{BES}}(u_{4,k}, u_{\bar{4},j}), \\
\left( \frac{x_{\bar{4},k}^+}{x_{\bar{4},k}^-} \right)^L &= \prod_{j=1}^{K_{\bar{4}}} \frac{u_{\bar{4},k} - u_{\bar{4},j} + i}{u_{\bar{4},k} - u_{\bar{4},j} - i} \prod_{j=1}^{K_1} \frac{1 - 1/x_{\bar{4},k}^-x_{1,j}}{1 - 1/x_{\bar{4},k}^+x_{1,j}} \prod_{j=1}^{K_3} \frac{x_{\bar{4},k}^- - x_{3,j}}{x_{\bar{4},k}^+ - x_{3,j}} \times \\
&\times \prod_{j \neq k}^{K_{\bar{4}}} \sigma_{\text{BES}}(u_{\bar{4},k}, u_{\bar{4},j}) \prod_{j=1}^{K_4} \sigma_{\text{BES}}(u_{\bar{4},k}, u_{4,j}),
\end{aligned} \tag{4.1}$$

where the spectrum of string energies is expressed in terms of the roots  $u_4$  and  $u_{\bar{4}}$ , which carry momentum, as follows:

$$E = h(\lambda) \mathcal{Q}_2, \tag{4.2}$$

the conserved charges being expressed in terms of the roots as

$$\mathcal{Q}_n = \sum_{j=1}^{K_4} \mathbf{q}_n(u_{4,j}) + \sum_{j=1}^{K_{\bar{4}}} \mathbf{q}_n(u_{\bar{4},j}), \quad \mathbf{q}_n = \frac{i}{n-1} \left( \frac{1}{(x^+)^{n-1}} - \frac{1}{(x^-)^{n-1}} \right). \tag{4.3}$$

The Zhukovsky variables are defined in terms of the roots as

$$x + \frac{1}{x} = \frac{u}{h(\lambda)}, \quad x^\pm + \frac{1}{x^\pm} = \frac{1}{h(\lambda)} \left( u \pm \frac{i}{2} \right). \tag{4.4}$$

Recalling that  $p_j = \frac{1}{i} \log \frac{x_{4,j}^+}{x_{4,j}^-}$  and  $\bar{p}_j = \frac{1}{i} \log \frac{x_{4,j}^+}{x_{4,j}^-}$ , we have

$$E = \sum_{j=1}^{K_4} \frac{1}{2} \left( \sqrt{1 + 16h(\lambda)^2 \sin^2 \frac{p_j}{2}} - 1 \right) + \sum_{j=1}^{K_{\bar{4}}} \frac{1}{2} \left( \sqrt{1 + 16h(\lambda)^2 \sin^2 \frac{\bar{p}_j}{2}} - 1 \right). \quad (4.5)$$

At large 't Hooft coupling we have

$$h(\lambda) \simeq \sqrt{\lambda/2}. \quad (4.6)$$

The rapidity variable expressed in terms of the momentum of the roots is given by

$$u_{4,j} = \frac{1}{2} \cot \left( \frac{p_j}{2} \right) \sqrt{1 + 16h(\lambda)^2 \sin^2 \left( \frac{p_j}{2} \right)^2}. \quad (4.7)$$

In the near plane wave limit, the BES kernel [2] reduces to the AFS phase factor [42]:

$$\sigma_{\text{AFS}}(u_j, u_k) = e^{i\theta_{jk}}, \quad (4.8)$$

where

$$\theta_{jk} = \sum_{r=2}^{\infty} h(\lambda) [\mathbf{q}_r(x_j) \mathbf{q}_{r+1}(x_k) - \mathbf{q}_r(x_k) \mathbf{q}_{r+1}(x_j)]. \quad (4.9)$$

The Bethe equations can be solved for the momenta  $p_j$  in the near plane wave limit only with a judicious perturbative Ansatz as

$$p_j = \frac{2\pi n_j}{J} + \frac{A}{J^2} + \frac{B\lambda'}{J^2} + \frac{C\lambda'^2}{J^2} + \frac{D\lambda'^3}{J^2} \dots, \quad (4.10)$$

where we solve order by order determining the expansion coefficients. Eventually we plug the solution for the momenta in the dispersion relation (4.5) to get the spectrum.

#### 4.1 Warm up: recap of the $SU(2) \times SU(2)$ subsector

Consider the cases  $(K_{u_4}, K_{u_{\bar{4}}}, K_{u_1}, K_{u_2}, K_{u_3}) = (2, 0, 0, 0, 0)$  and  $(K_{u_4}, K_{u_{\bar{4}}}, K_{u_1}, K_{u_2}, K_{u_3}) = (0, 2, 0, 0, 0)$ , which are clearly identical. Due to the level matching condition, we have only one independent momentum,  $p$ . Plugging the expansion (4.10) in the Bethe equations, one gets, up to order  $\lambda'^2$  and  $\frac{1}{J}$ :

$$\frac{1}{J} [A - 2\pi n + \lambda' (B + 8n^3 \pi^3) + \lambda'^2 (C - 32n^5 \pi^5) + \lambda'^3 (D + 192n^7 \pi^7)] = 0, \quad (4.11)$$

which completely determines the momentum up to the desired perturbative order. We have

$$A = 2n\pi, \quad B = -8n^3 \pi^3, \quad C = 32n^5 \pi^5, \quad D = -192n^7 \pi^7, \quad (4.12)$$

which plugged in the dispersion relation (4.5) gives the spectrum:

$$E_{20000} = 4n^2 \pi^2 \lambda' - 8n^4 \pi^4 \lambda'^2 + 32n^6 \pi^6 \lambda'^3 + \frac{1}{J} (8n^2 \pi^2 \lambda' - 64n^4 \pi^4 \lambda'^2 + 448n^6 \pi^6 \lambda'^3 - 3328n^8 \pi^8 \lambda'^4) \dots \quad (4.13)$$

which is the spectrum of the string states  $|s_{1,2}\rangle = \left(a_n^{1,2}\right)^\dagger \left(a_{-n}^{1,2}\right)^\dagger |0\rangle$ , addressed in [26].

Consider now the case  $(K_{u_4}, K_{u_{\bar{4}}}, K_{u_1}, K_{u_2}, K_{u_3}) = (1, 1, 0, 0, 0)$ . Similarly, due to the level matching condition, there is only one independent momentum,  $p$ . Yet we can build two different configurations, which shall be degenerate: the  $u_4$  root carrying momentum  $p$  and the  $u_{\bar{4}}$  carrying  $-p$ , or viceversa. The perturbative expansion of the Bethe equations reads:

$$\frac{1}{J} [A + \lambda' B + \lambda^2 (C + 16n^5 \pi^5) + \lambda'^4 (D + 128n^7 \pi^7)] = 0, \quad (4.14)$$

which gives

$$A = 0, \quad B = 0, \quad C = -16n^5 \pi^5, \quad D = -128n^7 \pi^7, \quad (4.15)$$

and therefore

$$E_{11000} = 4n^2 \pi^2 \lambda' - 8n^4 \pi^4 \lambda'^2 + 32n^6 \pi^6 \lambda'^3 - \frac{1}{J} (64n^6 \pi^6 \lambda'^3 - 768n^8 \pi^8 \lambda'^4) + \dots, \quad (4.16)$$

which is the spectrum of the string states  $|t_{1,2}\rangle = \left(a_n^{1,2}\right)^\dagger \left(a_{-n}^{2,1}\right)^\dagger |0\rangle$ , addressed in [26].

Actually the matching, which can be perturbatively checked at arbitrary high orders in  $\lambda'$ , between these solutions to the Bethe equations and the near plane wave spectrum of string states in the  $SU(2) \times SU(2)$  subsector, discussed in [26], has provided one of the earliest tests of the all loop asymptotic Bethe Ansatz proposed in [7].

## 4.2 How to deal with auxiliary roots

The one-magnon bosonic states correspond to the configurations:  $(K_{u_4}, K_{u_{\bar{4}}}, K_{u_1}, K_{u_2}, K_{u_3}) = (1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (1, 0, 1, 1, 1)$  and  $(1, 0, 1, 1, 1)$ . The one-magnon fermionic configurations are instead  $(K_{u_4}, K_{u_{\bar{4}}}, K_{u_1}, K_{u_2}, K_{u_3}) = (1, 0, 1, 0, 0), (0, 1, 1, 0, 0), (1, 0, 1, 1, 0), (0, 1, 1, 1, 0)$ . Out of these, 32 two-magnon states may be formed. We are interested in those having degenerate energies in the plane-wave limit, since they correspond to the string configurations we have studied in Section 3. In the boson-boson sector these are  $(K_{u_4}, K_{u_{\bar{4}}}, K_{u_1}, K_{u_2}, K_{u_3}) = (1, 1, 1, 1, 1), (2, 0, 1, 1, 1)$  and  $(0, 2, 1, 1, 1)$ ; in the fermion-sector these are  $(K_{u_4}, K_{u_{\bar{4}}}, K_{u_1}, K_{u_2}, K_{u_3}) = (1, 1, 2, 0, 0), (1, 1, 2, 1, 0), (1, 1, 2, 2, 0), (2, 0, 2, 0, 0), (2, 0, 2, 1, 0), (2, 0, 2, 2, 0), (0, 2, 2, 0, 0), (0, 2, 2, 1, 0)$  and  $(0, 2, 2, 2, 0)$ .

Taking into account the exact  $\mathbb{Z}_2$  degeneracy due to  $p \rightarrow -p$  symmetry and the double occurrence of  $(..210)$  and  $(..111)$  states due to branching of auxiliary roots we obtain 24 states having plane-wave degenerate spectrum, which exactly corresponds to the degenerate string two oscillator spectrum.

Below we therefore solve Bethe equations for these states and find their spectrum. We work at large  $\lambda$ , in the first order in  $\frac{1}{J}$ , and up to the 4th order in  $\lambda' = \frac{\lambda}{J^2}$ . The procedure is a perturbative expansion in  $\frac{1}{J}$  and  $\lambda'$ , parallel to the warm up exercise of the  $SU(2) \times SU(2)$  recalled above.

The order in  $\lambda'$  seems to be improvable *ad infinitum*; we chose the fourth order due to two considerations. First, it is order  $\lambda'^3$  where discrepancy between the spectrum of gauge invariant operators and near plane-wave string energies has been first found for the  $AdS_5/CFT_4$  correspondence, and cured with the introduction of the AFS phase factor [42] interpolating between weak and strong coupling. Thus, this feature having been

substantially inherited in the  $\text{AdS}_4/\text{CFT}_3$  correspondence, as one can read from the Bethe equations (4.1), agreement at  $\lambda'^4$  is such a nontrivial statement that be considered an all-order result. Second, larger values of the order become problematic (but not impossible) on *Mathematica*.

To regularize Bethe equations for those solutions carrying auxiliary roots  $u_i = 0, u_i = \infty$  perturbatively, one should add twist parameters  $\epsilon_{1,2,3}$ , as suggested in [49], in the following way:

$$\begin{aligned} e^{i\epsilon_1} &= \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + \frac{i}{2}}{u_{1,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{1 - 1/x_{1,k}x_{4,j}^+}{1 - 1/x_{1,k}x_{4,j}^-} \prod_{j=1}^{K_{\bar{4}}} \frac{1 - 1/x_{1,k}x_{\bar{4},j}^+}{1 - 1/x_{1,k}x_{\bar{4},j}^-}, \\ e^{i\epsilon_2} &= \prod_{j \neq k}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + \frac{i}{2}}{u_{2,k} - u_{1,j} - \frac{i}{2}} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}}{u_{2,k} - u_{3,j} - \frac{i}{2}}, \\ e^{i\epsilon_3} &= \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}}{u_{3,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k} - x_{4,j}^+}{x_{3,k} - x_{4,j}^-} \prod_{j=1}^{K_{\bar{4}}} \frac{x_{3,k} - x_{\bar{4},j}^+}{x_{3,k} - x_{\bar{4},j}^-}, \end{aligned}$$

The Bethe equations must be solved perturbatively for the momenta of the physical roots, in terms of the solution for the auxiliary roots expressed through the parameters  $\epsilon_{1,2,3}$ . At the end of the procedure, one takes the limit  $\epsilon_{1,2,3} \rightarrow 0$  and plugs the solution for the momenta in the dispersion relation (4.5). The spectrum of the boson-boson configurations, up to order  $\lambda'^4$ , is given in the following Table (9). In the table we show the quantity  $\varepsilon$ , defined as

$$E = \varepsilon_0 + \frac{\varepsilon}{J} + O\left(\frac{1}{J^2}\right). \quad (4.17)$$

In the “Note” column we show the final form of the lowest Bethe equation (the one for  $u_4$ ) that is being actually solved. The phase  $\sigma_{AFS}$  and spectral variable  $u(p)$  is meant as function of  $p$ , the latter given by (4.10). It is important to realize that these energy corrections are quite different from those for 20000, 11000 states (4.16), (4.13), despite the singularity of the roots. Auxiliary roots going to infinity (in the  $u$  plane) do not result in full cancellation of their respective contributions in the equations for  $u_4$  and  $u_{\bar{4}}$ , since the  $x(u)$  are different for these solutions. A solution of 20111 type with  $x_{1,2,3} = \infty$  would have been equivalent to 20000; however, in our case  $x_1 = \infty, x_2 = \infty, x_3 = 0$  which yields a solution of a totally different type due to extra  $x^+/x^-$  factor coming from the right-hand side of Bethe equation.

Similarly in the next Table (10) below we show the bifermionic part of the two-magnon sector of Bethe Ansatz,  $\varepsilon$  defined as above.

This table is quite remarkable, since all states presented here are also found on the string side, and the energies coincide up to the highest order done on the Bethe Ansatz side. Given this coincidence, as discussed above, this clearly points to an all-order equivalence for the finite-size corrections calculated from the Bethe Ansatz and from the string theory, in the limit  $\lambda' \rightarrow 0, J \rightarrow \infty$ , for all the two impurity light bosonic states. This is a remarkable further test of the Bethe Ansatz framework and a significant effort towards quantum integrability of strings in  $\text{AdS}_4 \times \mathbb{CP}_3$ .

**Table 9.** Boson-boson spectrum from Bethe Ansatz

state					Energy coefficient $\varepsilon$	Auxiliary roots	Note
$K_4$	$K_{\bar{4}}$	$K_3$	$K_2$	$K_1$			
1	1	1	1	1	$-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + 2048n^8\pi^8\lambda'^4$	$u_1 = \frac{1}{\epsilon_1}, x_1 \rightarrow \infty$ $u_2 = \frac{1}{\epsilon_2}, x_2 \rightarrow \infty$ $u_3 = \frac{1}{\epsilon_3}, x_3 \rightarrow 0$	$e^{ip(J+1)} = \sigma_{AFS}$ $J+1$ due to extra $x^-/x^+$ from $\prod^{K_3}(\dots)$
2	0	1	1	1	$-32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 - 2048n^8\pi^8\lambda'^4$	———— // ————	$e^{ip(J+1)} = \frac{2u+i}{2u-i}\sigma_{AFS}$
1	1	1	1	1	$-16n^2\pi^2\lambda' + 96n^4\pi^4\lambda'^2 - 768n^6\pi^6\lambda'^3 + 6144n^8\pi^8\lambda'^4$	$u_1 = \frac{1}{\epsilon_1}$ $u_2 = \frac{1}{\epsilon_2}$ $x_3 = \frac{1}{2\sqrt{2}n\pi\sqrt{\lambda'}}$	$e^{ipJ} = \frac{J+8i\pi^3\lambda'n^3-4i\pi n}{J}\sigma_{AFS}$
2	0	1	1	1	$-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + 2048n^8\pi^8\lambda'^4$	———— // ————	$e^{ipJ} = \frac{2u+i}{2u-i} \frac{J+8i\pi^3\lambda'n^3-4i\pi n}{J}\sigma_{AFS}$

To facilitate this comparison and summarize, let us represent the spectrum in a more concise form, showing the expansion coefficients in powers of  $\lambda'$  and the multiplicities of the states. For conciseness we do not write out the states 02..., since they are fully equivalent to the corresponding states 20.... All states 11... are twice degenerate to all orders due to  $n \rightarrow -n$  symmetry. Thus each of the states on the right hand side must be duplicated, which yields correct matching of the number of the degrees of freedom. Boson-boson sector is compared in the Table (11). Analogously, fermion-fermion spectrum comparison is done in the Table (12).

### 4.3 Claim to exactness

The spectacular coincidence of the  $\lambda'$  expansions for Bethe energies with the string energies supposes that it might be exact. This exactness can actually be seen directly in some of the cases. In the previous subsection the procedure to solve Bethe equations was to start with “highest” auxiliary nodes 1, 2, 3, then descend to the physical magnons 4,  $\bar{4}$ . Here we act reversely: start with the physical node, the momentum of which is known exactly in  $\lambda'$  form the exact string spectrum

$$\epsilon = \frac{4\pi^2 n^2 \lambda' \left( A - \frac{8\pi^2 (A+1)n^2 \lambda'}{8\pi^2 n^2 \lambda' + 1} \right)}{J}, \quad (4.20)$$

where  $A = 2, 0, -2, -4$  for the four admissible energy values of our spectrum. We can thus use the highest auxiliary node equation as a test. We have seen for several states of Bethe Ansatz (e.g. the (1, 1, 1, 1, 1), (2, 0, 1, 1, 1) states) that the first equation is non-trivially satisfied in a regular manner, that is, by a systematic improvement of the expansion one can satisfy the Bethe equation up to all orders.

**Table 10.** Fermion-fermion spectrum from Bethe Ansatz

state $K_4 \ K_{\bar{4}} \ K_3 \ K_2 \ K_1$					Energy coefficient $\varepsilon$	Auxiliary roots
1	1	2	2	0	$-32n^4\pi^4\lambda'^2 + 256n^6\pi^6\lambda'^3 - 2048n^8\pi^8\lambda'^4$	$u_{21} = \frac{1}{2}(u_{31} + u_{32} - i\sqrt{u_{31}^2 - 2u_{32}u_{31} + u_{32}^2 + 1})$ $u_{22} = \frac{1}{2}(u_{31} + u_{32} + i\sqrt{u_{31}^2 - 2u_{32}u_{31} + u_{32}^2 + 1})$ $x_{31} = -2i - \frac{i}{2n^2\pi^2\lambda'} + 2in^2\pi^2\lambda'$ $x_{32} = 2i + \frac{i}{2n^2\pi^2\lambda'} - 2in^2\pi^2\lambda'$
2	0	2	2	0	$8n^2\pi^2\lambda' - 96n^4\pi^4\lambda'^2 + 768n^6\pi^6\lambda'^3 - 6144n^8\pi^8\lambda'^4$	————//————
1	1	2	1	0	$-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + 2048n^8\pi^8\lambda'^4$	$u_2 = \frac{1}{2}(u_{31} + u_{32})$ $x_{31}, x_{32}$ solutions of $\frac{(x_3 - x(u+i/2))(x_3 - x(-u+i/2))}{(x_3 - x(u-i/2))(x_3 - x(-u-i/2))} = e^{i\epsilon_3}$
2	0	2	1	0	$-32n^4\pi^4\lambda'^2 + 256n^6\pi^6\lambda'^3 - 2048n^8\pi^8\lambda'^4$	————//————
1	1	2	1	0	$-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + 2048n^8\pi^8\lambda'^4$	$u_2 = \frac{1}{2}\left(u_{31} + u_{32} + \frac{4}{\epsilon_2}\right)$ $x_{31} = \frac{(1+4u(p)^2)\epsilon_3}{4\sqrt{2}J\sqrt{\lambda'}}$ , $u(p) = \frac{1}{2}\cot\left(\frac{p}{2}\right)\sqrt{1+2\lambda\sin^2\frac{p}{2}}$ $x_{32} = -\frac{(1+4u(p)^2)(-2i+\epsilon_3)}{4\sqrt{2}J\sqrt{\lambda'}}$
2	0	2	1	0	$-32n^4\pi^4\lambda'^2 + 256n^6\pi^6\lambda'^3 - 2048n^8\pi^8\lambda'^4$	————//————
1	1	2	0	0	$-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + 2048n^8\pi^8\lambda'^4$	$x_{31} = \frac{J\epsilon_3}{4\sqrt{2}n^2\pi^2\sqrt{\lambda'}}$ $x_{32} = \frac{J\epsilon_3}{4\sqrt{2}n^2\pi^2\sqrt{\lambda'}}$
2	0	2	0	0	$-32n^4\pi^4\lambda'^2 + 256n^6\pi^6\lambda'^3 - 2048n^8\pi^8\lambda'^4$	————//————

## 5 Conclusion

### 5.1 Summary of the near BMN calculations

The main results of this Paper can be summarized as follows:

- Our calculations provide a highly non-trivial test for the validity of the string Hamiltonian for three and four-particle interaction vertices in a near Penrose limit computed in [46].
- The one-loop correction to the single-magnon dispersion relation, as expected, is the same for bosonic and fermionic excitation, it is finite and exponentially small in  $J$

**Table 11.** Boson-boson spectrum comparison

Expansion coefficient	Multiplicity	Corresponding BA states	Corresponding ST states
$c_1$ $c_2$ $c_3$ $c_4$ $c_5$		$K_4$ $K_{\bar{4}}$ $K_3$ $K_2$ $K_1$	State nr.
0 $-\frac{1}{2}$ $\frac{1}{2}$ $-\frac{1}{2}$ $\frac{1}{2}$	2	2 0 1 1 $1_{\text{branch } 1}$	5, 7
$-1$ $\frac{1}{2}$ $-\frac{1}{2}$ $\frac{1}{2}$ $-\frac{1}{2}$	4	2 0 1 1 $1_{\text{branch } 2}$ 1 1 1 1 $1_{\text{branch } 1}$	2, 3, 6, 8
$-2$ $\frac{3}{2}$ $-\frac{3}{2}$ $\frac{3}{2}$ $-\frac{3}{2}$	2	1 1 1 1 $1_{\text{branch } 2}$	1, 4

(4.18)

**Table 12.** Fermion-fermion spectrum comparison

Expansion coefficient	Multiplicity	Corresponding BA states	Corresponding ST states
$c_1$ $c_2$ $c_3$ $c_4$ $c_5$		$K_4$ $K_{\bar{4}}$ $K_3$ $K_2$ $K_1$	State nr.
1 $-\frac{3}{2}$ $\frac{3}{2}$ $-\frac{3}{2}$ $\frac{3}{2}$	2	2 0 2 2 0	23, 24
0 $-\frac{1}{2}$ $\frac{1}{2}$ $-\frac{1}{2}$ $\frac{1}{2}$	8	1 1 2 2 0 2 0 2 1 $0_{\text{branch } 1}$ 2 0 2 1 $0_{\text{branch } 2}$ 2 0 2 0 0	9, 10, 17, 18, 19, 20, 21, 22
$-1$ $\frac{1}{2}$ $-\frac{1}{2}$ $\frac{1}{2}$ $-\frac{1}{2}$	6	1 1 2 1 $0_{\text{branch } 1}$ 1 1 2 1 $0_{\text{branch } 2}$ 1 1 2 0 0	11, 12, 13, 14, 15, 16

(4.19)

for  $J$  large. The regularization prescription implied by the cubic Hamiltonian and by consequent unitarity arguments, gives a vanishing one loop correction to the strong-weak coupling interpolating function  $h(\lambda)$ ,  $a_1 = 0$ .

- In the two-particle sector the finite-size corrections (the  $1/J$ -corrections) to magnon interaction energies on the string side and on the Bethe-Ansatz side are exactly the same up to the fourth order in  $\lambda' \equiv \frac{\lambda}{J^2}$ .

The second result in the one-particle sector relies on the argument about the unitarity-preserving property of the regularization. This argument, in view of its very general nature, should be applicable to all Bethe Ansatz states, yet it remains an open problem e.g. how exactly it would work for e.g. the GKP case [72], or spinning string states. However we may so far claim that a possible reason of the problems arising with the equal-frequency regularization (linear divergencies; disagreement between strings and the algebraic curve) is a possible unitarity violation by regularization. Thus the problems arising with it may be, somewhat loosely, called “unitarity anomaly”.



The significance of the third result is to establish another instance of the “mutual understanding” between the conjectured BA at all couplings with strings on  $AdS_4 \times CP^3$  in Penrose limit. The BA is asymptotic and thus is not *a priori* expected to work at strong coupling and arbitrary length. Yet it works, as established by our third result, not just asymptotically at  $J \rightarrow \infty$  but also at least at the order  $1/J$ . And at that order it is completely non-trivial that equivalence between the spectrum states holds at the fourth order in  $\lambda'$ .

We conjecture that the equivalence is actually an exact one, and extends towards higher orders in  $\frac{1}{J}$ .

## 5.2 Further questions

The full Bethe Ansatz, with *all* finite size and loop corrections, at both weak and strong coupling is encoded in the Y-system [31, 73]. For  $AdS_5/CFT_4$  this infinite system of functional equations has recently been shown to be equivalent to the T-system in [74], which is reducible to a finite number of integral equations. Thus of importance would be to test the results for finite-size corrections in strings against the T-system. The Lüscher terms are absent in the asymptotic Bethe Ansatz; string calculations would normally see them directly; in our case the exponentially-suppressed finite-size corrections look precisely like the typical Lüscher corrections do. Of extreme interest would be to compare in the one-loop sector the T-system with the direct Lüscher calculations (for a review see [50] for example), and with our string calculation of the finite size correction to the dispersion relation.

It is the strong coupling limit where the Y-system calculation should be easier to perform, since for strong coupling the functional/integral equations become algebraic and a full analytic solution becomes possible. Yet to our knowledge these solutions have so far been applied to GKP [72] states mostly, and not to BMN. Therefore this should be one of the major lines of further research - to obtain the self-energy finite-size corrections for the near-BMN spectrum directly from Y or T system, comparing them with the string calculation.

Probably the most urgent and straightforward further direction of the present work is, on the grounds of the same techniques, its extension to the computation of the finite size corrections of string states involving at least one heavy mode. Actually the Penrose limit of the geometry decouples light and heavy modes, having different dispersion relation. Yet they look equally fundamental, both being described by a Fourier series of harmonic-oscillator like modes, such that one might think that the fundamental degrees of freedom of the theory are  $8B + 8F$ . When dealing with finite size corrections, we have extensively discussed how infinite sums appear in the computation of the spectrum, which need to be regularized. The momentum conservation at the cubic light-light-heavy vertex forces the cutoff on the mode numbers of a heavy mode being twice as that of a light one. This is the first glimpse of the interpretation of a heavy mode being, rather than fundamental, a bound state of two light modes, such that indeed the basic degrees of freedom are  $4B+4F$ , as in the Bethe Ansatz framework. Actually the Bethe program gives a recipe for building a single light heavy oscillator state, being a composite of the fundamental roots. If this picture is

correct, the finite size spectrum of light-heavy or heavy-heavy two oscillator states must match the corresponding solutions of the Bethe equations. If this occurs, the puzzle about the interpretation of the heavy mode would be definitely solved.

The test of the  $AdS_5$  results at strong coupling has taken place at two loops for GKP strings, and at 1 loop for the near-BMN limit, yielding perfect agreement to the Y-system. Therefore the second-loop corrections to the self-energies and the first-loop corrections to scattering amplitudes could be interesting to calculate. Our Hamiltonian approach here would be extremely hard to implement, so we guess that perhaps the continuous Lagrangian field-theoretical approach [30] might be used provided it is supplemented by a unitarity-preserving regularization.

Our regularization for self-energies [44] is in essence equivalent to the one suggested by Gromov and Mikhaylov [71], who employ cutoffs, and for the sums consisting of heavy  $\omega^H$  and light  $\omega^L$  modes use the prescription

$$\text{Reg} \left( \sum \omega^H(n) + \omega^L(n) \right) \rightarrow \sum \left( \omega^H(n) + \omega^L \left( \frac{n}{2} \right) \right). \quad (5.1)$$

A similar regularization has been applied in the one- and two-spin BMN sector by Lipstein and Bandres [75]. An important finding of their paper is that the “equal-frequency” regularization leads to a linear divergence in the double-spin BMN string for the algebraic curve result, whereas the Gromov-Mikhaylov prescription yields all convergent results.

Let us mention here that in a very elucidating unpublished Note<sup>1</sup> by Gromov, Mikhaylov and Vieira a very general relation between the regularized one-loop self-energies from algebraic curve/Bethe Ansatz on one side and worldsheet string semiclassics on the other was derived. Gromov, Mikhaylov and Vieira show in the Note that if equal frequency (“world-sheet”) prescription is used then there are linear divergences in self-energies and inconsistency between higher charges of the integrable system as calculated from the discrete Bethe Ansatz; that would mean that the Bethe Ansatz equations must be modified in some way. On the other hand, if the unequal cutoff (“algebraic curve”) prescription is used, then no linear divergency arises and all charges are the same; Bethe Ansatz remains valid in the form we know it. We emphasize here that the result is stated in the Note by its authors as absolutely universal, extending thus greatly the double-spin BMN string obtained by Bandres and Lipstein [75]; this complies to our universal unitarity argument.

A special investigation is due on applicability of our regularization prescription for the one-particle sector beyond the near-BMN limit. This question is especially interesting with regard to the generic GKP strings/twist-2 gauge operators. Different subsectors of this sector include long and short spinning folded strings, rotating circular strings, with  $s$  large, very large or not very large (in each case a sophisticated technical definition of “largeness” or “smallness” is present). Certainly the physics is quite different from the BMN case. The S-matrix argument we use here should be clarified in adaptation to different sets of oscillations, since it was based on a BMN-spectrum. The spectrum of GKP oscillations, as obtained by Alday, Arutyunov and Bykov [69], contains of 6 bosons of zero mass, one boson of  $\sqrt{2}$  mass, one mass 2 boson and 6 mass 1 fermions. It is not clear therefore how

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<sup>1</sup>We thank Victor Mikhaylov for a clarifying discussion on this note.

the regularization (“unequal frequency”, “algebraic curve”) suggested by Astolfi, Grignani, Harmark and Orselli in [44], further argued for by Minahan and Zarembo, explored by Gromov and Mikhaylov [71] could be applied in its exact form<sup>2</sup> in terms of mode numbers on world-sheet. We conjecture that the unitarity argument will work here as well, although the explicit form of the argument based on the knowledge of certain pieces of the S-matrix would be essentially different.

Thus we wish to draw once more the attention to the unitarity issue in the regularization for generic sectors calling for further research into this subject. A hint may be a very interesting suggestion made by Gromov and Mikhaylov in [71]. Namely the “universal” prescription is to choose equal positions in the  $x$ -space for the excitations, where the  $x(u)$  algebraic curve coordinate. For all known cases, such as  $\text{AdS}_5$  and  $\text{AdS}_4$  this automatically leads to the correct mode structure and an “algebraic-curve” type prescription which satisfies the unitarity condition and is divergence-free. Thus a link, probably of a very general nature, must be established between this simple scheme and the world-sheet unitarity conservation.

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## References

- [1] N. Beisert and M. Staudacher, “Long-range  $PSU(2, 2|4)$  Bethe ansatz for gauge theory and strings,” *Nucl. Phys.* **B727** (2005) 1–62, [hep-th/0504190](#).
- [2] N. Beisert, B. Eden, and M. Staudacher, “Transcendentality and crossing,” *J. Stat. Mech.* **0701** (2007) P021, [hep-th/0610251](#).
- [3] N. Gromov, V. Kazakov, and P. Vieira, “Exact Spectrum of Anomalous Dimensions of Planar  $N=4$  Supersymmetric Yang-Mills Theory,” *Phys.Rev.Lett.* **103** (2009) 131601, [arXiv:0901.3753](#) [[hep-th](#)].
- [4] D. Bombardelli, D. Fioravanti, and R. Tateo, “Thermodynamic Bethe Ansatz for planar  $\text{AdS/CFT}$ : a proposal,” *J. Phys.* **A42** (2009) 375401, [arXiv:0902.3930](#) [[hep-th](#)].

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<sup>2</sup>We specially thank B.Basso for a discussion on this point.

- [5] G. Arutyunov and S. Frolov, “Thermodynamic Bethe Ansatz for the  $AdS(5) \times S(5)$  Mirror Model,” *JHEP* **0905** (2009) 068, [arXiv:0903.0141 \[hep-th\]](#).
- [6] N. Gromov, V. Kazakov, A. Kozak, and P. Vieira, “Exact Spectrum of Anomalous Dimensions of Planar  $N = 4$  Supersymmetric Yang-Mills Theory: TBA and excited states,” *Lett. Math. Phys.* **91** (2010) 265–287, [arXiv:0902.4458 \[hep-th\]](#).
- [7] N. Gromov and P. Vieira, “The all loop  $AdS_4/CFT_3$  Bethe ansatz,” *JHEP* **01** (2009) 016, [arXiv:0807.0777 \[hep-th\]](#).
- [8] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, “ $\mathcal{N} = 6$  superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” [arXiv:0806.1218 \[hep-th\]](#).
- [9] G. Arutyunov and S. Frolov, “Superstrings on  $AdS_4 \times CP^3$  as a Coset Sigma-model,” *JHEP* **09** (2008) 129, [arXiv:0806.4940 \[hep-th\]](#).
- [10] B. Stefanski, jr, “Green-Schwarz action for Type IIA strings on  $AdS_4 \times CP^3$ ,” *Nucl. Phys.* **B808** (2009) 80–87, [arXiv:0806.4948 \[hep-th\]](#).
- [11] D. Sorokin and L. Wulff, “Evidence for the classical integrability of the complete  $AdS_4 \times CP^3$  superstring,” *JHEP* **1011** (2010) 143, [arXiv:1009.3498 \[hep-th\]](#).
- [12] D. Sorokin and L. Wulff, “Peculiarities of String Theory on  $AdS_4 \times CP^3$ ,” [arXiv:1101.3777 \[hep-th\]](#).
- [13] A. Cagnazzo, D. Sorokin, and L. Wulff, “More on integrable structures of superstrings in  $AdS(4) \times CP(3)$  and  $AdS(2) \times S(2) \times T(6)$  superbackgrounds,” [arXiv:1111.4197 \[hep-th\]](#).  
\* Temporary entry \*.
- [14] J. A. Minahan and K. Zarembo, “The Bethe ansatz for superconformal Chern-Simons,” *JHEP* **09** (2008) 040, [arXiv:0806.3951 \[hep-th\]](#).
- [15] D. Gaiotto, S. Giombi, and X. Yin, “Spin Chains in  $N=6$  Superconformal Chern-Simons-Matter Theory,” *JHEP* **04** (2009) 066, [arXiv:0806.4589 \[hep-th\]](#).
- [16] D. Bak and S.-J. Rey, “Integrable Spin Chain in Superconformal Chern-Simons Theory,” *JHEP* **10** (2008) 053, [arXiv:0807.2063 \[hep-th\]](#).
- [17] C. Kristjansen, M. Orselli, and K. Zoubos, “Non-planar ABJM Theory and Integrability,” *JHEP* **03** (2009) 037, [arXiv:0811.2150 \[hep-th\]](#).
- [18] B. I. Zwiebel, “Two-loop Integrability of Planar  $N=6$  Superconformal Chern-Simons Theory,” *J. Phys.* **A42** (2009) 495402, [arXiv:0901.0411 \[hep-th\]](#).
- [19] J. A. Minahan, W. Schulgin, and K. Zarembo, “Two loop integrability for Chern-Simons theories with  $N=6$  supersymmetry,” *JHEP* **03** (2009) 057, [arXiv:0901.1142 \[hep-th\]](#).
- [20] D. Bak, H. Min, and S.-J. Rey, “Integrability of  $N=6$  Chern-Simons Theory at Six Loops and Beyond,” *Phys.Rev.* **D81** (2010) 126004, [arXiv:0911.0689 \[hep-th\]](#).
- [21] J. Minahan, O. Ohlsson Sax, and C. Sieg, “Magnon dispersion to four loops in the ABJM and ABJ models,” *J.Phys.A* **A43** (2010) 275402, [arXiv:0908.2463 \[hep-th\]](#).
- [22] J. Minahan, O. Ohlsson Sax, and C. Sieg, “Anomalous dimensions at four loops in  $N=6$  superconformal Chern-Simons theories,” *Nucl.Phys.* **B846** (2011) 542–606, [arXiv:0912.3460 \[hep-th\]](#).
- [23] F. Levkovich-Maslyuk, “Numerical results for the exact spectrum of planar  $AdS_4/CFT_3$ ,”

- [arXiv:1110.5869 \[hep-th\]](#). \* Temporary entry \*.
- [24] G. Grignani, T. Harmark, and M. Orselli, “The  $SU(2) \times SU(2)$  sector in the string dual of  $N=6$  superconformal Chern-Simons theory,” *Nucl. Phys.* **B810** (2009) 115–134, [arXiv:0806.4959 \[hep-th\]](#).
  - [25] G. Grignani, T. Harmark, M. Orselli, and G. W. Semenoff, “Finite size Giant Magnons in the string dual of  $N=6$  superconformal Chern-Simons theory,” *JHEP* **12** (2008) 008, [arXiv:0807.0205 \[hep-th\]](#).
  - [26] D. Astolfi, V. G. M. Puletti, G. Grignani, T. Harmark, and M. Orselli, “Finite-size corrections in the  $SU(2) \times SU(2)$  sector of type IIA string theory on  $AdS_4 \times CP^3$ ,” *Nucl. Phys.* **B810** (2009) 150–173, [arXiv:0807.1527 \[hep-th\]](#).
  - [27] T. McLoughlin, R. Roiban, and A. A. Tseytlin, “Quantum spinning strings in  $AdS_4 \times CP^3$ : testing the Bethe Ansatz proposal,” *JHEP* **11** (2008) 069, [arXiv:0809.4038 \[hep-th\]](#).
  - [28] P. Sundin, “The  $AdS_4 \times CP^3$  string and its Bethe equations in the near plane wave limit,” *JHEP* **02** (2009) 046, [arXiv:0811.2775 \[hep-th\]](#).
  - [29] K. Zarembo, “Worldsheet spectrum in  $AdS(4)/CFT(3)$  correspondence,” *JHEP* **0904** (2009) 135, [arXiv:0903.1747 \[hep-th\]](#).
  - [30] M. C. Abbott and P. Sundin, “The Near-Flat-Space and BMN Limits for Strings in  $AdS_4 \times CP^3$  at One Loop,” [arXiv:1106.0737 \[hep-th\]](#). \* Temporary entry \*.
  - [31] D. Bombardelli, D. Fioravanti, and R. Tateo, “TBA and Y-system for planar  $AdS(4)/CFT(3)$ ,” *Nucl. Phys.* **B834** (2010) 543–561, [arXiv:0912.4715 \[hep-th\]](#).
  - [32] J. M. Henn, J. Plefka, and K. Wiegandt, “Light-like polygonal Wilson loops in 3d Chern-Simons and ABJM theory,” *JHEP* **1008** (2010) 032, [arXiv:1004.0226 \[hep-th\]](#).
  - [33] M. S. Bianchi, M. Leoni, A. Mauri, S. Penati, C. A. Ratti, *et al.*, “From Correlators to Wilson Loops in Chern-Simons Matter Theories,” *JHEP* **1106** (2011) 118, [arXiv:1103.3675 \[hep-th\]](#). \* Temporary entry \*.
  - [34] W.-M. Chen and Y.-t. Huang, “Dualities for Loop Amplitudes of  $N=6$  Chern-Simons Matter Theory,” [arXiv:1107.2710 \[hep-th\]](#). \* Temporary entry \*.
  - [35] M. S. Bianchi, M. Leoni, A. Mauri, S. Penati, and A. Santambrogio, “Scattering Amplitudes/Wilson Loop Duality In ABJM Theory,” [arXiv:1107.3139 \[hep-th\]](#).
  - [36] R. R. Metsaev, “Type IIB Green-Schwarz superstring in plane wave Ramond- Ramond background,” *Nucl. Phys.* **B625** (2002) 70–96, [arXiv:hep-th/0112044](#).
  - [37] D. Berenstein, J. M. Maldacena, and H. Nastase, “Strings in flat space and pp waves from  $\mathcal{N} = 4$  super Yang Mills,” *JHEP* **04** (2002) 013, [hep-th/0202021](#).
  - [38] J. Callan, Curtis G. *et al.*, “Quantizing string theory in  $AdS_5 \times S^5$ : Beyond the pp- wave,” *Nucl. Phys.* **B673** (2003) 3–40, [arXiv:hep-th/0307032](#).
  - [39] J. Callan, Curtis G., T. McLoughlin, and I. Swanson, “Holography beyond the Penrose limit,” *Nucl. Phys.* **B694** (2004) 115–169, [arXiv:hep-th/0404007](#).
  - [40] N. Beisert, C. Kristjansen, and M. Staudacher, “The dilatation operator of  $\mathcal{N} = 4$  super Yang-Mills theory,” *Nucl. Phys.* **B664** (2003) 131–184, [hep-th/0303060](#).
  - [41] J. A. Minahan and K. Zarembo, “The Bethe-ansatz for  $\mathcal{N} = 4$  super Yang-Mills,” *JHEP* **03** (2003) 013, [hep-th/0212208](#).

- [42] G. Arutyunov, S. Frolov, and M. Staudacher, “Bethe ansatz for quantum strings,” *JHEP* **10** (2004) 016, [hep-th/0406256](#).
- [43] R. Hernandez and E. Lopez, “Quantum corrections to the string Bethe ansatz,” *JHEP* **07** (2006) 004, [arXiv:hep-th/0603204](#).
- [44] D. Astolfi, V. G. M. Puletti, G. Grignani, T. Harmark, and M. Orselli, “Finite-size corrections for quantum strings on  $AdS_4 \times CP^3$ ,” *JHEP* **1105** (2011) 128, [arXiv:1101.0004 \[hep-th\]](#). \* Temporary entry \*.
- [45] P. Sundin, “On the worldsheet theory of the type IIA  $AdS(4) \times CP(3)$  superstring,” *JHEP* **1004** (2010) 014, [arXiv:0909.0697 \[hep-th\]](#).
- [46] D. Astolfi, V. G. M. Puletti, G. Grignani, T. Harmark, and M. Orselli, “Full Lagrangian and Hamiltonian for quantum strings on  $AdS(4) \times CP^{*3}$  in a near plane wave limit,” *JHEP* **1004** (2010) 079, [arXiv:0912.2257 \[hep-th\]](#).
- [47] T. Nishioka and T. Takayanagi, “On Type IIA Penrose Limit and  $N=6$  Chern-Simons Theories,” *JHEP* **0808** (2008) 001, [arXiv:0806.3391 \[hep-th\]](#).
- [48] G. Grignani, T. Harmark, A. Marini, and M. Orselli, “New Penrose Limits and  $AdS/CFT$ ,” *JHEP* **06** (2010) 034, [arXiv:0912.5522 \[hep-th\]](#).
- [49] D. Volin, “String hypothesis for  $gl(n|m)$  spin chains: a particle/hole democracy,” [arXiv:1012.3454 \[hep-th\]](#). \* Temporary entry \*.
- [50] R. A. Janik, “Review of  $AdS/CFT$  Integrability, Chapter III.5: Lüscher Corrections,” [arXiv:1012.3994 \[hep-th\]](#).
- [51] I. Shenderovich, “Giant magnons in  $AdS(4) / CFT(3)$ : Dispersion, quantization and finite-size corrections,” [arXiv:0807.2861 \[hep-th\]](#).
- [52] C. Ahn, P. Bozhilov, and R. Rashkov, “Neumann-Rosochatius integrable system for strings on  $AdS(4) \times CP^{*3}$ ,” *JHEP* **0809** (2008) 017, [arXiv:0807.3134 \[hep-th\]](#).
- [53] M. C. Abbott and I. Aniceto, “Giant Magnons in  $AdS(4) \times CP^{*3}$ : Embeddings, Charges and a Hamiltonian,” *JHEP* **0904** (2009) 136, [arXiv:0811.2423 \[hep-th\]](#).
- [54] M. C. Abbott, I. Aniceto, and O. O. Sax, “Dyonic Giant Magnons in  $CP^3$ : Strings and Curves at Finite J,” *Phys. Rev.* **D80** (2009) 026005, [arXiv:0903.3365 \[hep-th\]](#).
- [55] D. Bombardelli and D. Fioravanti, “Finite-Size Corrections of the  $CP^3$  Giant Magnons: the Lüscher terms,” *JHEP* **07** (2009) 034, [arXiv:0810.0704 \[hep-th\]](#).
- [56] T. Lukowski and O. O. Sax, “Finite size giant magnons in the  $SU(2) \times SU(2)$  sector of  $AdS_4 \times CP^3$ ,” *JHEP* **12** (2008) 073, [arXiv:0810.1246 \[hep-th\]](#).
- [57] C. Ahn and P. Bozhilov, “Finite-size Effect of the Dyonic Giant Magnons in  $N=6$  super Chern-Simons Theory,” *Phys. Rev.* **D79** (2009) 046008, [arXiv:0810.2079 \[hep-th\]](#).
- [58] M. C. Abbott, I. Aniceto, and D. Bombardelli, “Quantum Strings and the  $AdS_4/CFT_3$  Interpolating Function,” *JHEP* **1012** (2010) 040, [arXiv:1006.2174 \[hep-th\]](#).
- [59] C. Ahn, M. Kim, and B.-H. Lee, “Quantum finite-size effects for dyonic magnons in the  $AdS_4 \times CP^3$ ,” *JHEP* **09** (2010) 062, [arXiv:1007.1598 \[hep-th\]](#).
- [60] M. C. Abbott, I. Aniceto, and D. Bombardelli, “Real and Virtual Bound States in Lüscher Corrections for  $CP^3$  Magnons,” [arXiv:1111.2839 \[hep-th\]](#). \* Temporary entry \*.
- [61] G. Papathanasiou and M. Spradlin, “The Morphology of  $N=6$  Chern-Simons Theory,” *JHEP*



- [0907](#) (2009) 036, [arXiv:0903.2548 \[hep-th\]](#).
- [62] R. Suzuki, “Giant Magnons on  $CP^{*3}$  by Dressing Method,” *JHEP* **0905** (2009) 079, [arXiv:0902.3368 \[hep-th\]](#).
  - [63] C. Kalousios, M. Spradlin, and A. Volovich, “Dyonic Giant Magnons on  $CP^3$ ,” *JHEP* **07** (2009) 006, [arXiv:0902.3179 \[hep-th\]](#).
  - [64] Y. Hatsuda and H. Tanaka, “Scattering of Giant Magnons in  $CP^{*3}$ ,” *JHEP* **1002** (2010) 085, [arXiv:0910.5315 \[hep-th\]](#).
  - [65] N. Beisert, V. Dippel, and M. Staudacher, “A novel long range spin chain and planar  $\mathcal{N} = 4$  super Yang-Mills,” *JHEP* **07** (2004) 075, [hep-th/0405001](#).
  - [66] N. Beisert, “The  $su(2|2)$  dynamic S-matrix,” [hep-th/0511082](#).
  - [67] M. Leoni, A. Mauri, J. Minahan, O. Sax, A. Santambrogio, *et al.*, “Superspace calculation of the four-loop spectrum in  $N=6$  supersymmetric Chern-Simons theories,” *JHEP* **1012** (2010) 074, [arXiv:1010.1756 \[hep-th\]](#).
  - [68] T. McLoughlin and R. Roiban, “Spinning strings at one-loop in  $AdS_4 \times CP^3$ ,” *JHEP* **12** (2008) 101, [arXiv:0807.3965 \[hep-th\]](#).
  - [69] L. F. Alday, G. Arutyunov, and D. Bykov, “Semiclassical Quantization of Spinning Strings in  $AdS(4) \times CP^{*3}$ ,” *JHEP* **0811** (2008) 089, [arXiv:0807.4400 \[hep-th\]](#).
  - [70] C. Krishnan, “ $AdS_4/CFT_3$  at One Loop,” *JHEP* **09** (2008) 092, [arXiv:0807.4561 \[hep-th\]](#).
  - [71] N. Gromov and V. Mikhaylov, “Comment on the Scaling Function in  $AdS_4 \times CP^3$ ,” *JHEP* **04** (2009) 083, [arXiv:0807.4897 \[hep-th\]](#).
  - [72] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” *Phys. Lett.* **B428** (1998) 105–114, [arXiv:hep-th/9802109](#).
  - [73] N. Gromov and F. Levkovich-Maslyuk, “Y-system, TBA and Quasi-Classical strings in  $AdS(4) \times CP^3$ ,” *JHEP* **1006** (2010) 088, [arXiv:0912.4911 \[hep-th\]](#).
  - [74] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, “Solving the  $AdS/CFT$  Y-system,” [arXiv:1110.0562 \[hep-th\]](#). \* Temporary entry \*.
  - [75] M. A. Bandres and A. E. Lipstein, “One-Loop Corrections to Type IIA String Theory in  $AdS(4) \times CP^3$ ,” *JHEP* **1004** (2010) 059, [arXiv:0911.4061 \[hep-th\]](#).
  - [76] O. Bergman and S. Hirano, “Anomalous radius shift in  $AdS(4)/CFT(3)$ ,” *JHEP* **0907** (2009) 016, [arXiv:0902.1743 \[hep-th\]](#).
  - [77] M. Bertolini, J. de Boer, T. Harmark, E. Imeroni, and N. A. Obers, “Gauge theory description of compactified pp-waves,” *JHEP* **01** (2003) 016, [hep-th/0209201](#).
  - [78] K. Sugiyama and K. Yoshida, “Type IIA string and matrix string on pp-wave,” *Nucl. Phys.* **B644** (2002) 128–150, [arXiv:hep-th/0208029](#).
  - [79] S.-j. Hyun and H.-j. Shin, “ $N = (4,4)$  type IIA string theory on pp-wave background,” *JHEP* **10** (2002) 070, [arXiv:hep-th/0208074](#).
  - [80] J. Gomis, D. Sorokin, and L. Wulff, “The complete  $AdS_4 \times CP^3$  superspace for the type IIA superstring and D-branes,” *JHEP* **03** (2009) 015, [arXiv:0811.1566 \[hep-th\]](#).
  - [81] P. A. Grassi, D. Sorokin, and L. Wulff, “Simplifying superstring and D-brane action in the  $AdS_4 \times CP^3$  superbackground,” *JHEP* **08** (2009) 060, [arXiv:0903.5407 \[hep-th\]](#).

- [82] R. R. Metsaev and A. A. Tseytlin, “Type IIB superstring action in  $\text{AdS}_5 \times S^5$  background,” *Nucl. Phys.* **B533** (1998) 109–126, [arXiv:hep-th/9805028](#).
- [83] R. Kallosh, J. Rahmfeld, and A. Rajaraman, “Near horizon superspace,” *JHEP* **09** (1998) 002, [arXiv:hep-th/9805217](#).
- [84] B. E. W. Nilsson and C. N. Pope, “HOPF FIBRATION OF ELEVEN-DIMENSIONAL SUPERGRAVITY,” *Class. Quant. Grav.* **1** (1984) 499.